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GREEN'S FUNCTIONS IN NONRENORMALIZABLE MASSLESS $\varphi^6_{3+\varepsilon}$ THEORY. PART II. REMOVAL OF THE CUTOFF

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Symanzik's method of renormalization is applied to construct Green's functions of the nonrenormalizable massless $\varphi^6_{3+\varepsilon}$ theory. The unmodified method leads to the expansion for vertex functions in four dimensions which contains logarithms and also square roots of the coupling constant. Higher terms in the quasi-perturbative expansion are analyzed.

1. *Strategy of the $\Lambda \to \infty$ limit*

In Part I of the present work we have formulated the regularized version of the massless scalar $\varphi^6_{3+\varepsilon}$, $\varepsilon > 0$, theory. The regularization was imposed in terms of the ultraviolet (UV) cutoff Λ, see (I.0.2) (in this part we are referring directly to the formulae of Part I). We have also limited ourselves to $\varepsilon < 2$. All these were intermediate steps in the effort to define Green's functions (GF) for the UV nonrenormalizable theory. We have also analyzed the differential vertex operation Λ DVO where Λ is the parameter with respect to which differentiation is performed.

Now we shall try to see how the vertex functions behave when $\Lambda \to \infty$. Following Symanzik, to study the $\Lambda \to \infty$ limit, we integrate the identity (I.4.1) with respect to Λ. For this purpose we write the identity (I.4.1) in the notation (I.4.3):

$$\frac{\partial}{\partial \Lambda} \Gamma^{2n}_{\Lambda} = \sum_{i=5}^{9} \Lambda^{2-a_i-\varepsilon(b_i-1)} c_i(g\mu^{-2\varepsilon}\Lambda^{2\varepsilon}, \varepsilon)\Gamma^{2n}_{\Lambda,i}, \tag{II.1.1}$$

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where we made use of (I.2.3) and (I.4.4a) and where, we recall, $a_5 = a_6 = 4$, $a_7 = a_8 = a_9 = 5$, $b_5 = 2$, $b_6 = 4$, $b_7 = 1$, $b_8 = 3$, $b_9 = 5$. Let us stress that this identity was derived in the (renormalized according to the standard subtractions) finite order perturbation theory and therefore after integration over the values of Λ which include $\Lambda = \infty$ can hold only for $\varepsilon < 1/L$ (cf. Section 1 of Part I). Hence, the integrated identity

$$\Gamma_\Lambda^{2n} = h_0^{2n} + \int_\infty^\Lambda d\lambda \sum_{i=5}^{9} \lambda^{2-a_i-\varepsilon(b_i-1)} c_i(g\mu^{-2\varepsilon}\lambda^{2\varepsilon}, \varepsilon)\Gamma_{\lambda,i}^{2n} \tag{II.1.2}$$

is valid only for perturbation orders $V < (n-1)/2+\varepsilon^{-1}$ (from (I.1.3) with (I.1.2)). The $h_0^{2n}(\ldots)$ is the integration constant corresponding to $\Gamma_\infty^{2n}(\ldots)$ in perturbation theory. The fact that the $\int_\infty^\Lambda d\lambda\ldots$ integral in (II.1.2) vanishes at $\Lambda = \infty$ follows from the fact that the integrand for the finite perturbation order decreases at $\lambda \to \infty$. This decrease follows from (I.4.4) with (I.0.3) as far as the $c_i(\ldots)$ are concerned and from the observation that for the $\Gamma_{\lambda,i}^{2n}$ a large-λ expansion analogous to (I.0.3) also holds. From comparison of (I.0.3) with (II.1.2), both at $\Lambda \to \infty$, we get that the $h_0^{2n}(\ldots)$ equals the Λ-independent term in (I.0.3). Namely, the double sum there can vanish at $\Lambda \to \infty$ only if $-1+\varepsilon k < 0$ which, in view of $k \leqslant L$, is in fact true for $\varepsilon < 1/L$.

Now we are going to write a formula for the VF analogous to (II.1.2) but which would be valid in the whole interval (0,2). This will be the crucial step because then the consequent formulae will allow us to prove that the Γ_Λ^{2n} exist also for $\Lambda = \infty$ and $\varepsilon \in (0, 2)$ if some simple conditions are satisfied. Roughly speaking, we shall indicate that at $\Lambda = \infty$ the r.h.s. of appropriately modified formula (II.1.2) can be finite but different from h_0^{2n} and ε-singularities of the h_0^{2n} will be cancelled by singularities of the remainder. The most natural way hereto is to continue analytically the r.h.s. of (I.1.2) from $\varepsilon < 1/L$. The "non-renormalizable" h_0^{2n} can have of course no worse singularities than poles at rational $\varepsilon \geqslant 1/L$ so that the analytic continuation will not affect its functional form and for $\varepsilon \in (0, 2)$ we write

$$\Gamma_\Lambda^{2n}(g, \mu, \varepsilon) = h_0^{2n}(g, \mu, \varepsilon) + \begin{pmatrix} \text{a.c.} \\ \text{from} \\ \varepsilon < \dfrac{1}{L} \end{pmatrix} \int_\infty^\Lambda d\lambda \sum_{i=5}^{9} \lambda^{2-a_i-\varepsilon(b_i-1)} c_i(g\mu^{-2\varepsilon}\lambda^{2\varepsilon}, \varepsilon)\Gamma_{\lambda,i}^{2n} \tag{II.1.3}$$

("a.c." stands for analytic continuation which, of course, can make sense only for finite perturbation order). The fact that we have to continue just from $\varepsilon < 1/L$ reminds us that also the evaluation prescriptions of Part I were perturbative, i.e. depending on the order considered. The analytic continuation in (II.1.3) is nontrivial because this can destroy the form of the integral as a matter of fact for any ε but for perturbation orders $V \geqslant (n-1)/2+\varepsilon^{-1}$. Therefore, it will be natural to try to represent the integral in (II.1.3) in the form of another analytic continuation from a region not depending on the perturbation order. This being done, the continuation from $\varepsilon < 1/L$ will become superfluous and consequenly formulae will not depend on the perturbation order. This means that our final results will concern the global properties of the GF and not apply only to the corresponding perturbative contributions. This feature will let us to discover, taking (II.1.3) as the starting point, such properties of the Γ^{2n}_{∞} which by no means could be seen in perturbation theory.

2. *Reintegration with respect to the cutoff*

For the analytic continuation of the integral $\int_{\infty}^{\Lambda} d\lambda \ldots$ in (II.1.3) we integrate by parts:

$$\int_{\infty}^{\Lambda} d\lambda \ldots = \int_{\infty}^{\Lambda} d\lambda \frac{\partial f_i}{\partial \lambda} \Gamma^{2n}_{\lambda,i} = f_i(\lambda)\Gamma^{2n}_{\lambda,i}\Big|^{\Lambda}_{\infty} - \int_{\infty}^{\Lambda} d\lambda \frac{\partial \Gamma^{2n}_{\lambda,i}}{\partial \lambda} f_i(\lambda),$$

specifically setting

$$f_i(\lambda) = \int_{\infty}^{\lambda} ds s^{2-a_i-\varepsilon(b_i-1)} c_i(g\mu^{-2\varepsilon}s^{2\varepsilon}, \varepsilon),$$

i.e. we write

$$\int_{\infty}^{\Lambda} d\lambda \ldots = \Gamma^{2n}_{\Lambda,i}(g, \mu, \varepsilon) \int_{\infty}^{\Lambda} d\lambda \lambda^{2-a_i-\varepsilon(b_i-1)} c_i(g\mu^{-2\varepsilon}\lambda^{2\varepsilon}, \varepsilon)$$

$$- \int_{\infty}^{\Lambda} d\lambda \frac{\partial \Gamma^{2n}_{\lambda,i}(g, \mu, \varepsilon)}{\partial \lambda} \int_{\infty}^{\lambda} ds s^{2-a_i-\varepsilon(b_i-1)} c_i(g\mu^{-2\varepsilon}s^{2\varepsilon}, \varepsilon). \tag{II.2.1}$$

In this Section we shall concentrate on the first term in (II.2.1), referring to the second one as to the higher term to be briefly discussed in Section 3.

$\Gamma^{2n}_{\Lambda,i}(g, \mu, \varepsilon)$ in the first term on the r.h.s. of (II.2.1) does not require continuation because it is singularity free, by construction (see Part I). Let us note that if for integration by parts $\int_{\Lambda}^{\lambda} ds \ldots$ were used instead of $\int_{\infty}^{\lambda} ds \ldots$ as above then we would now have $\Gamma^{2n}_{\infty,i}(g, \mu, \varepsilon)$ in place of $\Gamma^{2n}_{\Lambda,i}(g, \mu, \varepsilon)$ in (II.2.1).

For writing a continuation for the integral $\int_{\infty}^{\Lambda} d\lambda \ldots$, in the first term on the r.h.s. of (II.2.1) , we are in turns using analyticity of the $c_{ik}(\varepsilon)$ in (I.4.5), the fact that series there begins with $k_{\min 5} = \ldots = k_{\min 8} = 2$ and $k_{\min 9} = 3$ and the (assumed) analyticity of the $c_i(z, \varepsilon)$. Under the additional assumption that

$$\int_{0}^{\infty} d\lambda \lambda^{2-a_i-\varepsilon(b_i-1)} [c_i(g\mu^{-2\varepsilon}\lambda^{2\varepsilon}, \varepsilon) - 2i\delta_{i7}] = 0 \tag{II.2.2}$$

we can represent the integral in question as

$$-i\delta_{i7}\Lambda^{-2}+\mu^{-3-a_i-\varepsilon(b_i-1)}g^{(a_i-3)/2\varepsilon+(b_i-1)/2}$$

$$\times\left\{\sum_{k=k_{\min i}}^{\infty}[3-a_i+\varepsilon(2k+1-b_i)]^{-1}c_{ik}(\varepsilon)+\frac{1}{2\varepsilon}S_i(g\mu^{-2\varepsilon}\Lambda^{2\varepsilon},\varepsilon)\right\}, \qquad \text{(II.2.3)}$$

where

$$S_i(g\mu^{-2\varepsilon}\Lambda^{2\varepsilon},\varepsilon)$$

$$=\begin{pmatrix}\text{a.c.}\\ \text{from}\\ \varepsilon>1\end{pmatrix}\int_1^{g\mu^{-2\varepsilon}\Lambda^{2\varepsilon}} dz\, z^{\frac{1}{2}[(3-a_i)/\varepsilon-(1+b_i)]}[c_i(z,\varepsilon)-2i\delta_{i7}]. \qquad \text{(II.2.4)}$$

The expression (II.2.3) contributes to $\Gamma_\Lambda^{2n}(g,\mu,\varepsilon)$ and therefore the singularities at rational values of ε, as displayed in (II.2.3), cannot appear in $\Gamma_\Lambda^{2n}(g,\mu,\varepsilon)$ which was argued to be nonsingular for $\varepsilon\in(0,2)$.

Let us anticipate here that expressions like (II.2.3) with the characteristic factor $\mu^{-3-a_i-\varepsilon(b_i-1)}g^{(a_i-3)/2\varepsilon+(b_i-1)/2}$ included, will appear within analogous analysis for $\Gamma_{\Lambda,i}^{2n}(g,\mu,\varepsilon)$ which multiplies the presently discussed integral and identical factors will also appear for all $\Gamma_{\Lambda,i,j,\ldots}^{2n}(g,\mu,\varepsilon)$. This observation, together with an extension to the higher terms in (II.2.1), makes it possible to organize the expansion for $\Gamma_\Lambda^{2n}(\ldots)$ in terms of these characteristic powers of μ and g. However, in this paper we will write such an expansion only for the $\Lambda=\infty$ case. Let us also mention that subsequently taking expansions for $\Gamma_{\Lambda,i,j,\ldots}^{2n}(\ldots)$ (with increasing α) into account (when expanding $\Gamma_\Lambda^{2n}(\ldots)$) shows, in contradistinction to what the formula (II.2.3) alone suggests, that the functions of Λ of the type (II.2.4) will contribute also to the residua of $\Gamma_\Lambda^{2n}(\ldots)$ at rational values of ε. Hence, the Λ-independent parts of these residua can cancel against corresponding residua in the $h_0^{2n}(\ldots)$ while the Λ-dependent parts must mutually cancel among themselves.

Already it is seen that in the $\Lambda\to\infty$ limit one can get finite corrections to $h_0^{2n}(g,\mu,\varepsilon)$ only if the integral

$$\int_1^{\infty} dz\, z^{\frac{1}{2}[(3-a_i)/\varepsilon-(1+b_i)]}[c_i(z,\varepsilon)-2i\delta_{i7}]$$

converges in a subinterval of (1, 2). If so, the analyticity of $c_i(z,\varepsilon)$ being assumed, we shall be able to define the analytic continuation for all $\varepsilon\in(0,2)$. Namely, we can then write

$$\Gamma_\infty^{2n}(g,\mu,\varepsilon)=h_0^{2n}(g,\mu,\varepsilon)+\sum_{i=5}^{9}\Bigg\{\mu^{3-a_i-\varepsilon(b_i-1)}g^{(a_i-3)/2\varepsilon+(b_i-1)/2}$$

$$\times\left[\sum_{k=k_{\min i}}^{\infty}((3-a_i)+\varepsilon(2k+1-b_i))^{-1}c_{ik}(\varepsilon)+\frac{1}{2\varepsilon}S_i(\varepsilon)\right]\Gamma_{\infty,i}^{2n}(g,\mu,\varepsilon)+\text{higher terms}\Bigg\},$$

(II.2.5)

where

$$S_i(\varepsilon) = \begin{pmatrix} \text{a.c.} \\ \text{from} \\ \varepsilon > 1 \end{pmatrix} \int_1^{\infty} dz z^{\frac{1}{2}[(3-a_i)/\varepsilon-(1+b_i)]}[c_i(z, \varepsilon)-2i\delta_{i7}] \tag{II.2.6}$$

are the first five Symanzik's constants for the $\varphi^6_{3+\varepsilon}$ theory which were promised in the Introduction (see Part I). These constants are not expected to be plagued with ε-singularities but would be of course infinite if for $c_i(z, \varepsilon)$ the truncated expansions (I.4.5) were substituted. Let us notice that if we assumed that the $\int_0^{\infty} d\lambda \ldots$ integrals of (II.2.2) were just finite and not necessarily equal to zero then their constant values would have to be added to Symanzik's constants (II.2.6). In formula (II.2.5) it is of course understood that for the $\Gamma^{2n}_{\infty,i}(\ldots)$ analogous expression to that which was obtained above for Γ^{2n}_{∞} should be substituted.

3. *Expansion for vertex functions in the $\Lambda = \infty$ case*

From what has been said up to now we have seen that the information about Λ DVO for VF with multiple insertions is necessary for studying the Λ-dependence of $\Gamma^{2n}_{\Lambda}(\ldots)$ itself. It can be shown (see the Appendix) that

$$\frac{\partial}{\partial\lambda}\Gamma^{2n}_{\lambda,i} = \sum_{k=5}^{9} \lambda^{2-a_k-\varepsilon(b_k-1)} c_k(g\mu^{-2\varepsilon}\lambda^{2\varepsilon}, \varepsilon)\Gamma^{2n}_{\lambda,i,k}$$
$$+\sum_{l} r_{li}(g\mu^{-2\varepsilon}\lambda^{2\varepsilon}, \varepsilon)\lambda^{2-a_l-\varepsilon(b_l-1)}\Gamma^{2n}_{\lambda,l} \tag{II.3.1}$$

analogously for $\Gamma^{2n}_{\lambda,i,k}$, $i = 5, \ldots, 9$, $k = 5, \ldots, 9$ and so on. Hence, in the higher term

$$\int_{\infty}^{\Lambda} d\lambda \frac{\partial \Gamma^{2n}_{\lambda,i}}{\partial\lambda} \int_{\infty}^{\lambda} ds s^{2-a_i-\varepsilon(b_i-1)} c_i(g\mu^{-2\varepsilon}s^{2\varepsilon}, \varepsilon) \tag{II.3.2}$$

which has appeared in (II.2.1) we can insert (II.3.1) and it will be advantageous to write (II.3.2) in the form

$$\int_{\infty}^{\Lambda} d\lambda \sum_{k=5}^{9} \lambda^{2-a_k-\varepsilon(b_k-1)} f_{ki}(\lambda, \varepsilon)\Gamma^{2n}_{\lambda,i,k} + \int_{\infty}^{\Lambda} d\lambda \sum_{l} \lambda^{2-a_l-\varepsilon(b_l-1)} g_{li}(\lambda, \varepsilon)\Gamma^{2n}_{\lambda,l} \tag{II.3.3}$$

with

$$f_{ki}(\lambda, \varepsilon) = c_k(g\mu^{-2\varepsilon}\lambda^{2\varepsilon}, \varepsilon) \int_{\infty}^{\lambda} ds s^{2-a_i-\varepsilon(b_i-1)} c_i(g\mu^{-2\varepsilon}s^{2\varepsilon}, \varepsilon),$$

$$g_{li}(\lambda, \varepsilon) = r_{li}(g\mu^{-2\varepsilon}\lambda^{2\varepsilon}, \varepsilon) \int_{\infty}^{\lambda} ds s^{2-a_i-\varepsilon(b_i-1)} c_i(g\mu^{-2\varepsilon}s^{2\varepsilon}, \varepsilon), \tag{II.3.4}$$

because both terms in this formula are now analogous to the Λ-dependent part on the r.h.s. of (II.1.2) and from here on identical reasonings to those leading from (II.1.2) to

(II.2.5) can be simply repeated, with c_i replaced by f_{ki} and g_{li}. A next-to-higher term

$$\int_{\infty}^{\Lambda} d\lambda \frac{\partial \Gamma^{2n}_{\lambda,i,k}}{\partial \lambda} \int_{\infty}^{\lambda} ds s^{2-a_i-\varepsilon(b_i-1)} f_{ki}(s,\varepsilon) \tag{II.3.5}$$

will appear thereby from integration by parts and so on. This is the reason why in Part I we were also checking the analyticity properties of VF with arbitrarily many insertions. The procedure of taking into account the Λ DVO for the VF with a growing number of insertions reveals at $\Lambda = \infty$ additional nonperturbative constants similar to the constants (II.2.6). They arise because of the second term in (II.3.1). Thus, one can speak about the $\Lambda = \infty$ case only if one assumes that all these constants (the Symanzik constants) are finite.

The expressions at $\Lambda = \infty$ for the VF with insertions differ from (II.2.5) and from each other merely by obvious changes in notation. In particular,

$$\begin{aligned} \Gamma^{2n}_{\infty,i} = h^{2n}_{0i} + \sum_{j=5}^{9} \Big\{ \mu^{3-a_j-\varepsilon(b_j-1)} g^{(a_j-3)/2\varepsilon+(b_j-1)/2} \\ \times \Big[\sum_{k=k_{\min j}} (3-a_j+\varepsilon(2k+1-b_j))^{-1} c_{jk}(\varepsilon) + \frac{1}{2\varepsilon} S_j(\varepsilon) \Big] \Gamma^{2n}_{\infty,i,j} \Big\} \\ + \sum_{l} \Big\{ \mu^{3-a_l-\varepsilon(b_l-1)} g^{(a_l-3)/2\varepsilon+(b_l-1)/2} \\ \times \Big[\sum_{k=k_{\min l}} (3-a_l+\varepsilon(2k+1-b_l))^{-1} c_{lk}(\varepsilon) + \frac{1}{2\varepsilon} S_{li}(\varepsilon) \Big] \Gamma^{2n}_{\infty,l} \Big\} + \{\text{higher terms}\} \end{aligned} \tag{II.3.6}$$

for $i = 5, \ldots, 9$, analogously for $\Gamma^{2n}_{\infty,i,j}$ and so on. When such expressions will be substituted in (II.2.5) then it will become clear that also the singular parts of the corrections to $h_0^{2n}(\ldots)$ at $\Lambda = \infty$ must contain the nonperturbative constants. To conclude, in Symanzik's approach one can establish by perturbation theoretical methods merely the locations of ε-poles in separate contributions to the $\Gamma^{2n}_{\Lambda}(g, \mu, \varepsilon)$.

The observation that the $\mu^{3-a_l} g^{(a_l-3)/2\varepsilon}$ factors appear systematically in the above procedure allows us to reorganize the expansion for Γ^{2n}_{∞} (which up to now was given in terms of a growing number of insertions) accordingly and argue that

$$\Gamma^{2n}_{\infty}(g, \mu, \varepsilon) = \sum_{j=0}^{\infty} \mu^{-j} g^{j/2\varepsilon} h^{2n}_j(g, \mu, \varepsilon), \tag{II.3.7}$$

where

$$h^{2n}_j(g, \mu, \varepsilon) = \sum_{l=0}^{\infty} (\mu^{-\varepsilon} g^{\frac{1}{2}})^l h^{2n}_{jl}(\varepsilon) \tag{II.3.8}$$

for $j > 0$ and the $j = 0$ term is the $h_0^{2n}(\dots)$ of before. The fact that $h_j^{2n}(g, \mu, \varepsilon), j > 0$, is the half-integer power series in the coupling constant (and not integer power series as in $\varphi^4_{4+\varepsilon}$ theory) follows from the observation that among the factors $\mu^{-\varepsilon(b_i-1)}g^{(b_i-1)/2}$, except for those with odd b_7, b_8 and b_9, also those with b_5 and b_6 which are even occur and $b_5 = 2$. $h_j^{2n}(g, \mu, \varepsilon), j > 0$, now contain (products of) Symanzik's constants and all other similar constants.

In all of the discussions up to now we kept ε generic, but the expansion (II.3.7) is expected to be free of UV ε-singularities for arbitrary $\varepsilon \in (0, 2)$. Hence, the singularities appearing in $\Gamma^{2n}_\infty(\dots)$ at rational values of ε must cancel, giving rise to logarithms of the coupling constant. The physically interesting case of four dimensions ($\varepsilon = 1$) will be treated as the special case of ε rational.

In order to illustrate how the logarithms arise at $\varepsilon = 1$ let us consider, as an example, a contribution to h_j^{2n}, $j > 0$, of the type $(\mu^{-\varepsilon}g^{1/2})^l(\varepsilon-1)^{-1}$. This gives a contribution to Γ^{2n}_∞ of the form $\mu^{-j-\varepsilon l}g^{j/2\varepsilon+l/2}(\varepsilon-1)^{-1}$. This singular term can be cancelled (at $\varepsilon = 1$) by a term precisely equal to $-\mu^{-N}g^{N/2}(\varepsilon-1)^{-1}$ with $N = j+l$. Such a term can originate from h_0^{2n} or from another $h_j^{2n}, j > 0$, if $j+l$ is even but must stem from another $h_j^{2n}, j > 0$, if $j+l$ is an odd number because the h_0^{2n} can only have integer powers of g. One can easily find that

$$\lim_{\varepsilon\to 1} \left[(\varepsilon-1)^{-1}(\mu^{-j-\varepsilon l}g^{j/2\varepsilon+l/2} - \mu^{-j-l}g^{(j+l)/2})\right] \sim \mu^{-j-l}g^{(j+l)/2}\ln(g^{j/2}\mu^{-l}).$$

Hence, in four dimensions the singularities are cancelled, leaving behind the logarithms of the coupling constant, i.e. the expansion (II.3.7) reduces to

$$\Gamma^{2n}_\infty(g, \mu, \varepsilon = 1) = h_0^{2n}(g, \mu, \varepsilon = 1) + \sum_{j=1}^{\infty} (\log g)^j H_j^{2n}(g, \mu, \varepsilon = 1). \qquad \text{(II.3.9)}$$

The functions $H_j^{2n}(g, \mu, \varepsilon = 1)$ can be the power series in the square roots of the coupling constant, as the $h_j^{2n}(g, \mu, \varepsilon)$ are for $j > 0$ and ε generic. At the present time we do not see any reasons why the h_{jl}^{2n} of (II.3.8) with $j+l$ odd should have no finite parts at $\varepsilon = 1$ nor why the cancellations of singularities for the $h_{jl}^{2n}(\varepsilon)$ with l odd could not take place.

Outlook

One could expect that when original Symanzik's methods will be applied to give meaning to Green's functions of the massless $\varphi^6_{3+\varepsilon}$, $\varepsilon > 0$, theory then this will lead only nonsignificant modifications to compare with the known results for the massless $\varphi^4_{4+\varepsilon}$, $\varepsilon > 0$, theory. However, the main stream of the arguments (i.e. the Λ DVO with coefficients determined from the renormalization conditions in the regularized theory) did not exclude the odd powers of the cutoff Λ in the large-Λ expansion for the vertex functions of the regularized theory as well as the half integer powers of the coupling constant in the expansions (II.3.7) and (II.3.9) for vertex functions of the nonrenormalizable massless $\varphi^6_{3+\varepsilon}$ theory. These two features and IR difficulties in (I.4.4) constitute the main difference between $\varphi^6_{3+\varepsilon}$ theory and $\varphi^4_{4+\varepsilon}$ theory. From the technical details of our exposition it is clear that these differences would not be encountered if the coefficients c_5 and c_6 of the cutoff differential identity (I.4.2) were zero. However, results of explicit calculations for lowest order contributions to c_5 and c_6 (from (I.4.4b) and (I.4.4c)) are different from zero and we do not see any reasons why these coefficients should vanish.

The main achievement of Symanzik's approach to the problem of removal of divergences in Green's functions of nonrenormalizable quantum field theory so far is the reduction of a "renormalization of the nonrenormalizable" to a problem of existence of improper one-dimensional integrals (Symanzik's constants). For the norenormalizable massless $\varphi^6_{3+\varepsilon}$ theory the simplest of these are given by the formula (II.2.6). Hence, the whole problematics is open anew and a lot of work will have to be done to determine whether Symanzik's approach is satisfactory or not. In particular, all of the discussions above are far from being complete until some essentially nonperturbative methods will be invented. Within Symanzik's framework such nonperturbative methods are to be applied for checking the consistency of some of the assumptions (see e.g. (II.2.2)) and for computation of expressions like (II.2.6) (Symanzik's constants). Another urgent problem is to generalize Symanzik's approach to massive field theories in order to avoid infrared difficulties.

APPENDIX

The derivation of the Λ DVO (I.4.1) *and* (II.3.1)

We shall start with Lowenstein's differential vertex operation [10], [11],

$$\Lambda \frac{\partial}{\partial \Lambda} \Gamma^{2n}_{\Lambda, i, \dots, j} = \sum_k \frac{\partial c_k}{\partial \Lambda} \Gamma^{2n}_{\Lambda, i, \dots, j, [l_k]_3} \tag{1}$$

for the VF with an arbitrary number of insertions with zero momentum transfer to the graphs. In this formula c_k and $[l_k]_3$ are defined by the renormalized Lagrangian L_r of the theory, namely

$$L_r = -i \sum_k c_k [l_k]_3, \tag{2}$$

where $[l_k]_3$ are strictly operator parts of the Lagrangian L_r, i.e. all coefficients (apart from the combinatorial ones) are included in c_k. The brackets $[\ ]_3$ denote that the operators l_k have ascribed degree 3 (whereas their canonical dimensions can be different from 3). In our case, l_k are equal to O_1, O_2, O_3, O_4, O_7 in notation (I.2.1).

In formula (1) it is assumed that all operators l_k have ascribed degree three, whereas in the formulae (I.2.1), (II.3.1) l_k are supposed to have degree equal to their canonical dimensions. The appropriate change of the degrees of the operators l_k can be achieved by making use of Zimmermann's identities [9], [11], [12]. For VF with only one insertion these identities read:

$$\Gamma^{2n}_{\Lambda, [\varphi \Box^2 \varphi]_3} = \sum_{i=1}^{9} a_i \Gamma^{2n}_{\Lambda, O_i},$$

$$\Gamma^{2n}_{\Lambda,[\varphi^2]_3} = \sum_{i=1}^{4} b_i \Gamma^{2n}_{\Lambda,O_i},$$

$$\Gamma^{2n}_{\Lambda,[\varphi^4]_3} = \sum_{i=1}^{4} e_i \Gamma^{2n}_{\Lambda,O_i}, \tag{3}$$

where the coefficients a_i, b_i, e_i can be determined from the renormalization conditions (I.3.5). The identities (3) can be written in the general form

$$\Gamma^{2n}_{\Lambda,[O_s]_3} = \sum_{i=1}^{9} a_i^s \Gamma^{2n}_{\Lambda,O_i}. \tag{4}$$

Substituting the identities (3) into the Λ DVO (1) with $\alpha = 0$ one obtains formula (I.4.1).

Let us mention that in the first of the formulae (3) the integrated normal product $[\varphi\Box^2\varphi]_3$ is undersubtracted because it has degree $3 < \dim \varphi\Box^2\varphi$. However, this does not lead to difficulties because the propagator in graphs is regularized according to the Lagrangian (I.0.2).

The Zimmermann identities for VF with more than one insertion are much more complicated and we do not quote here these rather lengthy formulae (see Ref. [12]). We mention only that the general form of this identity for VF with two insertions is

$$\Gamma^{2n}_{\Lambda,[O_s]_d,O_k} = \sum_{i=1}^{9} a_i^s \Gamma^{2n}_{\Lambda,O_i,O_k} + \sum_{l} r_l^s \Gamma^{2n}_{\Lambda,O_l}, \tag{5}$$

where on the r.h.s. all operators have ascribed degrees equal to their canonical dimensions and a_i^s are the same as in (4). The operators O_l are scalar operators with dimensions satisfying the inequalities:

$$\dim O_s + \dim O_k - 3 \geqslant \dim O_l > d + \dim O_k - 3 \quad \text{if} \quad d < \dim O_s,$$

$$d + \dim O_k - 3 \geqslant \dim O_l > \dim O_s + \dim O_k - 3 \quad \text{if} \quad d > \dim O_s.$$

The formula (II.3.1) is obtained by substituting the identities (5) to the Λ DVO (1) with $\alpha = 1$ and by dimensional analysis for r_k^s analogous to that leading to the formula (I.4.2) for c_i.

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