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EXACT, MULTIPLE SOLITON SOLUTIONS FOR POLYNOMIAL FIELD THEORIES

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Exact solutions of nonlinear generalizations of the Klein Gordon equation analogous to multi-solitons of classical theories are constructed. The number of distinct solutions of this type is shown to be dependent upon the dimensionality of space-time. Some of the solutions are localized in 3+1 dimensions and are time dependent generalizations of single peak confined solutions.

In the past few years a remarkable number of exact solutions of nonlinear field equations have been found. The field equations studied have been diverse—ranging from Korteweg–de Vries and nonlinear Schrödinger which are first order in time to sine Gordon and nonlinear Klein Gordon which are second order in time. The variety of solutions include single and multi-peaked functions with plane symmetry—solitons—and static and time dependent solutions with spherical symmetry—instantons [1].

In this paper we discuss multi-peaked localized and semi-localized solutions of the nonlinear Klein Gordon equation

$$\partial_\mu \partial^\mu \Phi + m^2 \Phi + \lambda \Phi^3 = 0. \tag{1}$$

The number of dimensions is arbitrary. These solutions are time dependent generalizations of Goldstone solutions [2] and various solutions appropriate for confined systems [3]. Solutions of field equations with more general polynomial and non-polynomial interactions will be discussed elsewhere [4].

The method of constructing multiple soliton solutions of Eq. (1) is a special case of the method of base equations in which a solution of a nonlinear partial differential equation is found in terms of solutions of a linear partial differential equation [5]. The method is applicable to equations with polynomial or nonpolynomial interactions and in any dimensional space-time [6].

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A solution of Eq. (1) is

$$\Phi = u(1-\lambda u^2/8m^2)^{-1} = uD^{-1}, \tag{2}$$

where u is a solution of the Klein Gordon equation

$$\partial_\mu \partial^\mu u + m^2 u = 0 \tag{3}$$

subject to the restriction

$$\partial_\mu u \partial^\mu u + m^2 u^2 = 0. \tag{4}$$

The proof is by differentiation [5]. A useful identity is

$$\begin{aligned} u\partial_\mu D^{-p} &= 2p(\partial_\mu u)D^{-p-1}(\lambda u^2/8m^2) = -2p(\partial_\mu u)D^{-p-1}(-1+1-(\lambda u^2/8m^2)) \\ &= 2pD^{-p-1}\partial_\mu u - 2p(\partial_\mu u)D^{-p}. \end{aligned} \tag{5}$$

Using Eq. (5) one has

$$\partial_\mu \Phi = -(\partial_\mu u)D^{-1} + 2(\partial_\mu u)D^{-2} \tag{6}$$

and

$$\begin{aligned} \partial_\mu \partial^\mu \Phi = &-(\partial_\mu \partial^\mu u)D^{-1} + 2(\partial_\mu \partial^\mu u)D^{-2} - 2(\partial_\mu u)(\partial^\mu u)u^{-1}(D^{-2} - D^{-1}) \\ &+ 8(\partial_\mu u)(\partial^\mu u)(\lambda u/8m^2)D^{-3}. \end{aligned} \tag{7}$$

Using Eqs (3)—(4) one has

$$\partial_\mu \partial^\mu \Phi = m^2\Phi - 2m^2 uD^{-2} + 2m^2 uD^{-2} - 2m^2 uD^{-1} - \lambda\Phi^3 = -m^2\Phi - \lambda\Phi^3, \tag{8}$$

which is the desired result.

An n soliton solution for Φ can be constructed by letting

$$u_n = \sum_{i=1}^{n} a_i e^{\alpha_i k_i \cdot x}, \tag{9}$$

where

$$\alpha_i = (-m^2/k_i^2)^{1/2}, \quad (m^2 \neq 0,\ k_i^2 \neq 0), \tag{10}$$

$$k_i \cdot x = k_{i0}x_0 - \vec{k}_i \cdot \vec{x}. \tag{11}$$

The a_i are arbitrary constants while the four vectors k_i, in order to satisfy Eq. (4), must satisfy

$$m^2 + k_i \cdot k_j \alpha_i \alpha_j = 0. \tag{12}$$

Two specific conditions implied by Eq. (12) are of sufficient importance to display at the outset. These are

$$k_i \neq -k_j, \quad \vec{k}_i \cdot \vec{k}_j \neq 0 \quad (\text{any } i, j). \tag{13}$$

In addition to restricting the relative magnitudes of the components of the set $\{k_i\}$, Eq. (12) also limits the total number of different $\alpha_i k_i$ and thus the multiplicity of the solitons. The number of components of n vectors is $3n$, while the number of pairs $(i \neq j)$ in Eq. (12) is $n(n-1)/2$, so the system is not overdetermined if

$$3n \geqslant n(n-1)/2, \tag{14}$$

$$n \leqslant 7. \tag{15}$$

Hence, the maximum number of distinct functions $e^{\alpha_i k_i \cdot x}$ occurring in u_n of Eq. (9) is 7. Of course, the arbitrary constants a_i can be adjusted to obtain different forms for u_n, but these forms will contain no more than 7 different k_i. More generally, if the dimensionality of the space is p, Eq. (14) becomes

$$n_p \leqslant 2p-1. \tag{16}$$

However, in 1+1 space, Eq. (12) is even more restrictive. In this case it is easy to see that Eqs (12)—(13) imply

$$k_i/k_{i0} = k_j/k_{j0} \tag{17}$$

and further that the algebraic signs of the corresponding components of k_i and k_j are identical.

From Eq. (17) it is easily seen that the arguments of all the functions in Eq. (9) are identical in 1+1 space, i.e.

$$\arg_i = (-m^2/k_i^2)^{1/2} k_i \cdot x = (-m^2 k_{j0}^2/k_{i0}^2 k_j^2)^{1/2} (k_{i0}/k_{j0})(k_j \cdot x) = \arg_j. \tag{18}$$

Thus, in 1+1 space there is only one solution of the type given in Eq. (2) and Eqs (9)—(13). The solution may be localized or nonlocalized, depending on the signs of m^2/λ and $-m^2/k^2$. A particular case is given below.

In spaces of higher dimensionality, Eq. (12) can be solved for the direction cosines of the angles between $\vec{k}_i$ and $\vec{k}_j$. The result is

$$\cos\theta_{ij} = v_i v_j \pm [(v_i^2-1)(v_j^2-1)]^{1/2}, \tag{19}$$

where

$$v_i = k_i^0/|\vec{k}_i|. \tag{20}$$

To insure that $\cos\theta_{ij}$ lies between -1 and 1, the phase velocities v_i must satisfy restrictions determined by whether the k_i are timelike or spacelike. The basic inequality may be written

$$-1 - v_i v_j \leqslant \pm[(v_i^2-1)(v_j^2-1)]^{1/2} \leqslant 1 - v_i v_j. \tag{21}$$

From inspection of the center term in the inequality it is evident that to insure $\cos\theta_{ij}$ real all k_i must be spacelike or all timelike.

Consider first all k_i timelike, so all $|v_i| > 1$ (with no loss of generality, take all $k_i^0 > 0$). The right side of Eq. (21) is negative, so only the minus square root is acceptable. The inequality may be reversed to give

$$v_i v_j + 1 \geqslant [(v_i^2-1)(v_j^2-1)]^{1/2} \geqslant v_i v_j - 1. \tag{22}$$

Squaring, one finds

$$0 \geqslant (v_i - v_j)^2 \tag{23}$$

which is nonsense unless $v_i = v_j$, all i, j. But this requires from Eq. (19) that $\cos\theta_{ij} = 1$ (recalling that for timelike k only the minus square root in Eq. (19) is acceptable), hence all $\vec{k}_i$, $\vec{k}_j$ are parallel. That is, the solutions are effectively $1+1$ dimensional and, as shown above, reduce to a function of a single variable[1].

The case of special physical interest occurs for all k_i spacelike, so all $|v_i| < 1$. The interest in this case stems from the fact that the solutions are (semi) localized and non-singular for $\lambda < 0$ and $m^2 > 0$ — that is, for Goldstone-Higgs types of theories. The 1 soliton, by suitable choice of a in Eq. (9), reduces to the well known single lump solution

$$\Phi = (-2m^2/\lambda)^{1/2} \operatorname{sech} \alpha k \cdot x. \tag{25}$$

For arbitrary spacelike k_i in the general case the inequality in Eq. (9) is satisfied for all $|v_i| < 1$. It is clearly possible to transform to a Lorentz frame in which the time component of one k_i, say k_1, vanishes. Thus, the n soliton contains a static lump which is localized along the $\vec{k}_1$ direction. Furthermore, from Eq. (19) $\cos\theta_{ij}$ becomes

$$\cos\theta_{ij} = \pm(1-v_j^2)^{1/2}. \tag{26}$$

Now, if all $\vec{k}_j$ are parallel to $\vec{k}_1$, Eq. (26) requires all v_j to vanish and once again, from the arguments given in the $1+1$ dimensional case, the n soliton reduces to a 1 soliton. Perhaps the solutions given in Eqs (2), (9)—(13) should be called "collapsons" to describe this property.

For $n \geqslant 3$ it is possible to choose a linearly independent set of $\vec{k}_i$ in $3+1$ dimensions. However, the restriction, Eq. (13), excludes the solution with mutually orthogonal $\vec{k}_i$ (it is easy to see that the expressions for $\cos\theta_{ij}$ are incompatible for this case). A 3 soliton with non orthogonal $\vec{k}_i$ which is localized and non singular for $\lambda < 0$, $m^2 > 0$, is

$$\Phi_3 = \sum_{i=1}^{3} \Phi_{3i}, \tag{27}$$

$$\Phi_{3i} = a_i e^{\alpha_i k_i \cdot x} D^{-1} = a_i e^{\psi_i} D^{-1} \tag{28}$$

with

$$D = 1 - (\lambda/8m^2)\left(\sum_{i=1}^{3} a_i e^{\psi_i}\right)^2. \tag{29}$$

The linear independence of k_i insures that the ψ_i are independent variables for arbitrary x. Evidently, as ψ_i varies over its range $(-\infty < \psi_i < \infty)\Phi_{3i}$ is bounded. Furthermore, as ψ_j $(i \neq j)$ varies over its range Φ_{3i} reduces at most to a bounded function of the other two variables[2].

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[1] A similar argument can be made for any pair k_i, k_j for which the sign of the time components is opposite, hence $v_i < -1$, $v_j > 1$. The conclusion in this case is $|v_i| = |v_j|$, implying $\vec{k}_i$ antiparallel to $\vec{k}_j$.

[2] Pursuing this argument one can show that the solutions in Eq. (27) consist of incoming waves at all $k_i \cdot x = -\infty$. These waves propagate along $\vec{k}_i$, emerging at $k_i \cdot x = +\infty$, $k_j \cdot x$ finite, as outgoing waves with a phase shift. This is the property expected of multiple solitons. For complete details see Ref. [4].

The restriction in Eq. (13) leads to the result that all the multi-soliton functions described by Eqs (2), (9)—(13) are time dependent for all k_i spacelike. That is, a Lorentz transformation can take the function to a frame in which at most one k_i has vanishing time component. This lump is the remnant of the single soliton state, so in quantum field theory the n solitons may be regarded as time dependent expectation values of non-linear excited states of this ground state.

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