A QUASI-CLASSICAL DESCRIPTION OF ISOSPIN-CONSERVATION IN MULTIPARTICLE PRODUCTION

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(Received December 8, 1977)

Assuming that in a high energy collision n pions are created, which carry a total isospin $I = 0$, we develop a simple method for calculating the distribution over the different charge states. The method, although approximate, can reproduce some features of distributions and correlations of charged and neutral pions observed in experiment. In principle the method can also be used to calculate the distribution of the total number of created pions and of the neutral pions from the observed distribution of charged particles.

1. Introduction

From previous theoretical work [1]–[3] it is known that the incorporation of isospin conservation in multiparticle production can in principle explain a number of features of the distributions and correlations of charged and neutral pions. In particular it has become clear that the linear increase of the width of the charged particle distribution is, at least partly, due to this I-spin conservation. In all papers, so far, special assumptions about the production process had to be made, however. E.g. in reference [1] the final pions were described by a coherent state, which is a way to formulate independent emission. In references [2] and [3] it was assumed that the spatial part of the pionic wave function is completely symmetric. Reference [3] gives a dynamical model with which it was possible to reproduce most of the available experimental data and to predict in particular correlations between charged and neutral pions.

In all previous work, however, it was not clear whether the special dynamical assumption or the isospin conservation was most responsible for the agreement with experiment.

For that reason we would like in this paper to describe a method which enables us to incorporate the isospin conservation in a simple way, so that dynamical effects can be studied separately.

In a three-dimensional (isospin) space we consider a path which starts at the origin

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and consists of n unit steps. This n will be the total number of particles and the vector from the origin to the endpoint of the path, will be identified with the isospin vector I. For the total isospin to be zero we must therefore have a closed path. Now, in a full quantum mechanical treatment, starting from a n particle state with definite values I and I_3 for the total isospin and its third component, one can, by adding a single pion, construct nine different states with isospin $I-1$, I or $I+1$ and third component I_3-1, I_3 or I_3+1. In order to mimic this situation we will restrict each step of our path to be one of the six which are parallel or antiparallel to the I_1, I_2 and I_3-axis. A closed path therefore consists of an even number of steps in a simple cubic lattice.

The artifice of an even number of particles we do not consider as an essential shortcoming and could perhaps be overcome. The number of steps parallel to the I_1, I_2 and I_3-axis we call $2m_1$, $2m_2$ and $2m_3$ respectively. The set of non-negative integers (m_1, m_2, m_3) does not completely specify the path, but they will fix the number of charged and of neutral pions.

To each path with given (m_1, m_2, m_3) we now assign an a priori probability which is proportional to the number of different paths which have the same value for these three numbers. Clearly this probability is equal to

$$P(m_1, m_2, m_3) = N(n)\binom{2m_1}{m_1}\binom{2m_2}{m_2}\binom{2m_3}{m_3}, \tag{1}$$

where we choose the normalisation constant $N(n)$ is such a way that

$$\sum_{m_1m_2m_3}^{(m)} P(m_1, m_2, m_3) = P(n) \tag{2}$$

is the distribution of the total number of pions $n = 2m$. The summation is restricted to those triples (m_1, m_2, m_3) for which $m_1+m_2+m_3 = m$. The function $P(n)$ is an arbitrary and normalised distribution and contains all the information about the production process. Once it is known the distributions of and correlations between charged and neutral pions can be calculated, as will be shown later.

Introducing the abbreviation

$$A_k = 2^{-2k}\binom{2k}{k} \tag{3}$$

the following relations can easily be proved by complete induction:

$$\left.\begin{aligned} \sum_k A_k &= 2(l+1)A_{l+1} && (4)\\ \sum_k kA_k &= \tfrac{2}{3}\, l(l+1)A_{l+1} && (5)\\ \sum_k k^2A_k &= \tfrac{2}{15}\, l(l+1)(3l+2)A_{l+1} && (6) \end{aligned}\right\} k = 0, 1, 2, \ldots, l.$$

$$\left.\begin{aligned} \sum_{k_1k_2} A_{k_1}A_{k_2} &= 1 && (7)\\ \sum_{k_1k_2} k_1k_2A_{k_1}A_{k_2} &= \tfrac{1}{8}\, l(l-1) && (8) \end{aligned}\right\} k_1+k_2 = l.$$

Using Eqs (7) and (4) it immediately follows from (2) that $N(n)$ is given by

$$N(n) = N(2m) = \frac{2^{-n}P(n)}{(n+2)A_{m+1}}. \tag{9}$$

We now define the relation between the sets (m_1, m_2, m_3) and the numbers n_c and n_0 of charged and neutral pions as follows:

$$n_c = m_1 + m_2 + 2m_3 \tag{10}$$

and

$$n_0 = m_1 + m_2. \tag{11}$$

Notice that $n_c + n_0$ is indeed equal to the total number of particles. In words the relations (10) and (11) mean that for each step parallel to the I_3-axis a charged particle is added, while for each step in the $I_1 - I_2$-plane either a charged or a neutral pion is added with equal probability.

In a strict quantum mechanical sense the relations (10) and (11) cannot be correct, since in a representation in which I^2 and I_3 are diagonal it is impossible for n_c and n_0 to have definite values, because they do not commute with I^2. For large numbers, however, it may not be a bad approximation. We will try to make the rules (10) and (11) plausible by considering the operators for the isospin of the meson field in terms of creation and annihilation operators $a_i^*(\vec{p})$ and $a_i(\vec{p})$, i.e.,

$$I_j = -i\varepsilon_{jkl} \int_{\vec{p}} a_k^*(\vec{p}) a_l(\vec{p}). \tag{12}$$

For convenience of writing we will from now on omit the momentum $\vec{p}$. The operators for charged and neutral pions are $a_\pm^* = \frac{1}{\sqrt{2}}(a_1^* \pm i a_2^*)$ and $a_0^* = a_3^*$. The third component of the isospin becomes

$$I_3 = a_+^* a_+ - a_-^* a_-. \tag{13}$$

The other two components can also be written in terms of $a_\pm^*$ and a_0^*. It is, however, more instructive to introduce the boson operators

$$b_\pm^* = \frac{1}{\sqrt{2}}(a_2^* \pm i a_3^*) = \frac{i}{\sqrt{2}}\left(\pm a_0^* - \frac{1}{\sqrt{2}} a_+^* + \frac{1}{\sqrt{2}} a_-^*\right) \tag{14}$$

and

$$c_\pm^* = \frac{1}{\sqrt{2}}(a_3^* \pm i a_1^*) = \frac{1}{\sqrt{2}}\left(a_0^* \pm \frac{i}{\sqrt{2}} a_+^* \pm \frac{i}{\sqrt{2}} a_-^*\right). \tag{15}$$

The first two components of the isospin can then be written as

$$I_1 = b_+^* b_+ - b_-^* b_- \tag{16}$$

and

$$I_2 = c_+^* c_+ - c_-^* c_-. \tag{17}$$

From Eq. (13) we now see that a step in the positive I_3 direction is to be interpreted as adding a π^+. A step in the negative I_3-direction is equivalent to the addition of a π^-. A step in the positive (negative) I_1-direction is seen as the addition of a pion of the type $b_+(b_-)$. In both cases such a b-meson is either a π^0 or a π^+ or a π^- with a probability for a π^0 equal to the probability for a charged pion. For the steps parallel to the I_2-axis we have the same interpretation and this then explains our rules (10) and (11). We must mention, however, that we have only used plausibility arguments and that for instance the (quantum) effects of the phase factors in the formulae (14) and (15) are completely lost. Also, as pointed out previously and discussed in the last section, the number of available steps, hence the number of available states in our model is somewhat smaller than in the quantum mechanical calculations [7].

From the distribution (1) we can now calculate the probability to find n_c charged and n_0 neutral pions, by summing over m_1, m_2 and m_3 with the restrictions (10) and (11). We obtain

$$P(n_c, n_0) = \frac{A_{n_c - \frac{1}{2}n} P(n)}{(n+2)A_{m+1}}, \tag{18}$$

where $n = n_c + n_0 = 2m$.

We close this section by observing that from (10) and (11) it follows that there are never more π^0's than charged pions, or equivalently, that at least half of the produced pions are charged.

In the next section we will discuss some of the general properties of our distributions. In the third section we will then show how $P(n)$ can be obtained from a measurement of the distribution of charged particles.

2. *General properties*

Since our rules (10) and (11) are "derived" from a walk on a cubic lattice and from the formulas (13), (16) and (17), which also do not single out a special axis, we are convinced that in our method all three types of pions are treated on equal footing. This is partly confirmed by the fact that for the average values of n_c and n_0 we find

$$\bar{n}_c = \tfrac{2}{3}\bar{n} \quad \text{and} \quad \bar{n}_0 = \tfrac{1}{3}\bar{n}. \tag{19}$$

By using Eqs (5) and (7) it is indeed easily checked that

$$\bar{m}_1 = \bar{m}_2 = \bar{m}_3 = \tfrac{1}{6}\bar{n}, \tag{20}$$

from which Eq. (19) follows immediately. The average of the total number of particles is defined as

$$\bar{n} = \sum_{n=0,2,4,\dots} nP(n). \tag{21}$$

Dispersions and correlations can be calculated from the formulas

$$\overline{m_1^2} = \overline{m_2^2} = \overline{m_3^2} = \tfrac{1}{20}\,\overline{n^2}+\tfrac{1}{15}\,\bar{n} \tag{22}$$

and

$$\overline{m_1 m_2} = \overline{m_2 m_3} = \overline{m_1 m_3} = \tfrac{1}{60}\,\overline{n^2}-\tfrac{1}{30}\,\bar{n} \geqslant 0, \tag{23}$$

which can easily be proved using Eqs (4)—(8). For the dispersion D_c of the distribution of charged particles we then obtain

$$D_c^2 = \overline{n_c^2}-\bar{n}_c^2 = \tfrac{7}{15}\,D^2+\tfrac{1}{45}\,\bar{n}^2+\tfrac{1}{15}\,\bar{n}, \tag{24}$$

in which D is the dispersion of the total distribution. It is seen that even in the case where this latter distribution is very narrow, the charge dispersion D_c increases linearly with $\bar{n}_c = \frac{2}{3}\bar{n}$. In fact, for large $\bar{n}_c$ we then have

$$D_c \simeq 0.224\bar{n}_c \tag{25}$$

which is already almost half of the measured value [4]

$$D_c(\pi^-\mathrm{p}) \simeq (0.54\pm0.02)\bar{n}_c-(0.40\pm0.07) \tag{26}$$

and

$$D_c(\mathrm{pp}) \simeq (0.58\pm0.01)\bar{n}_c-(0.56\pm0.01). \tag{27}$$

The correlations $f_{cc} = \overline{n_c(n_c-1)}-\bar{n}_c^2$, $f_{00} = \overline{n_0(n_0-1)}-\bar{n}_0^2$ and $f_{c0} = \overline{n_c n_0}-\bar{n}_c\bar{n}_0$ can also be calculated using Eqs (22) and (23). They can be expressed in terms of D^2 or, with the help of Eq. (24), in terms of D_c^2. In addition to the trivial relation

$$f_{cc} = D_c^2-\bar{n}_c \tag{28}$$

we find

$$f_{00} = \tfrac{1}{45}\,D^2+\tfrac{1}{45}\,\bar{n}^2-\tfrac{4}{15}\,\bar{n} = \tfrac{1}{21}\,D_c^2+\tfrac{1}{21}\,\bar{n}_c^2-\tfrac{17}{42}\,\bar{n}_c \tag{29}$$

and

$$f_{c0} = \tfrac{1}{5}\,D^2-\tfrac{1}{45}\,\bar{n}^2-\tfrac{1}{15}\,\bar{n} = \tfrac{3}{7}\,D_c^2-\tfrac{1}{14}\,\bar{n}_c^2-\tfrac{1}{7}\,\bar{n}_c. \tag{30}$$

For high energies the latter is positive if and only if

$$D_c > 0.41\bar{n}_c,\ \text{i.e.}\ D > \tfrac{1}{3}\,\bar{n}. \tag{31}$$

Substitution of the experimental relations (26) and (27) leads to the following predictions of f_{00} and f_{c0} for the π^-p and pp cases

$$f_{00}(\pi^-\mathrm{p}) = (0.061\pm0.001)\bar{n}_c^2-(0.425\pm0.004)\bar{n}_c+(0.0076\pm0.003), \tag{32}$$

$$f_{00}(\mathrm{pp}) = (0.064\pm0.001)\bar{n}_c^2-(0.436\pm0.001)\bar{n}_c+(0.015\pm0.001), \tag{33}$$

$$f_{c0}(\pi^-\mathrm{p}) = (0.05\pm0.01)\bar{n}_c^2-(0.33\pm0.03)\bar{n}_c+(0.07\pm0.02), \tag{34}$$

$$f_{c0}(\mathrm{pp}) = (0.073\pm0.005)\bar{n}_c^2-(0.42\pm0.02)\bar{n}_c+(0.134\pm0.005). \tag{35}$$

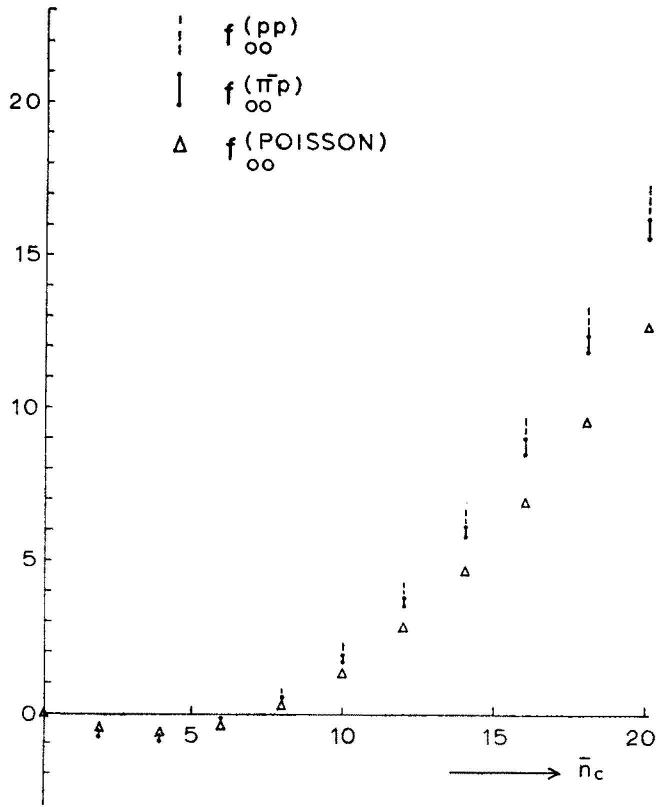

Fig. 1. Plots of correlation functions between two neutral pions as given by Eqs (32), (33) and (36)

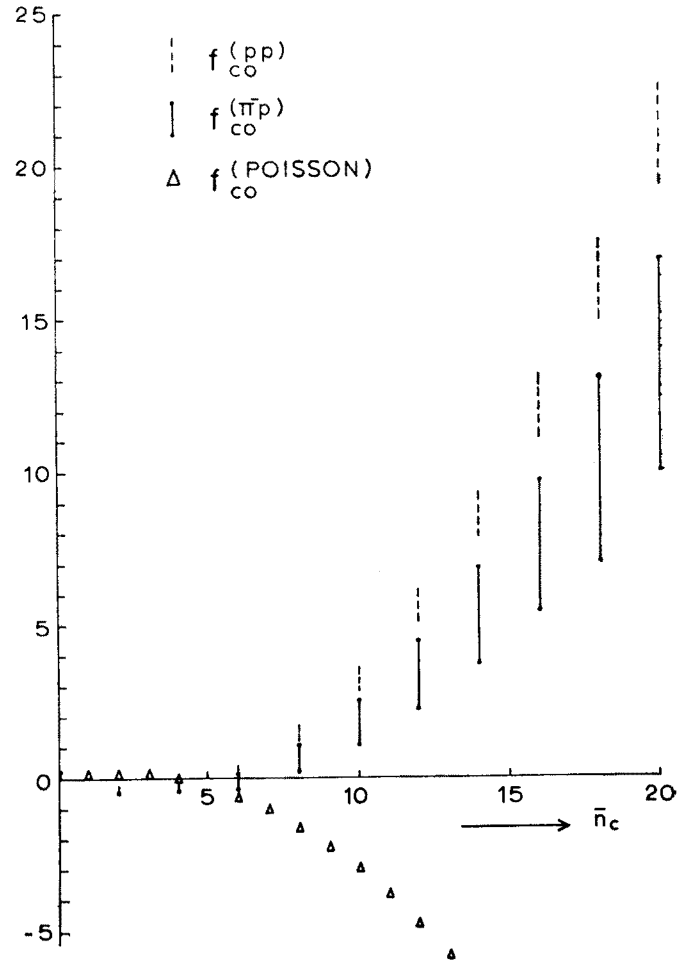

Fig. 2. Plots of correlation functions between one charged and one neutral pion as given by Eqs (34), (35) and (37)

For a Poisson-like distribution of the total number of particles the square of the dispersion is $D^2 = \bar{n}$. Via (24), (29) and (30) this leads to

$$f_{00}(P) = \tfrac{1}{20}\,\bar{n}_c^2 - \tfrac{11}{30}\,\bar{n}_c \tag{36}$$

and

$$f_{c0}(P) = -\tfrac{1}{20}\,\bar{n}_c^2 + \tfrac{1}{5}\,\bar{n}_c. \tag{37}$$

For the three cases (π^-p), (pp) and (Poisson) the functions f_{00} and f_{c0} are shown in figures 1 and 2.

3. Distributions

For a given distribution $P(n)$ of the total number of particles it is clear from Eq. (18) that the distribution of charged or of neutral particles can be obtained by summing over n_0 or n_c respectively. Since n_0 is never greater than n_c the charge distribution is represented by a finite sum

$$P_c(n_c) = \sum_k B(n_c, k)P(2n_c - 2k), \tag{38}$$

where the sum extends over all non-negative integers not larger than $\frac{1}{2}\,n_c$ and where the coefficients are given by

$$B(n_c, k) = \frac{A_k}{2(n_c+1-k)A_{n_c+1-k}}. \tag{39}$$

From Eq. (38) and the behaviour of $B(n_c, k)$ for large n_c and k, viz.

$$B(n_c, k) \simeq \frac{1}{2\sqrt{k(n_c+1-k)}}, \tag{40}$$

it is easy to show that if $P(n)$ satisfies KNO-scaling, so does $P_c(n_c)$. A special choice for $P(n)$ is the Poisson-like distribution

$$P(n) = \frac{1}{\cosh \tilde{n}} \cdot \frac{\tilde{n}^n}{n!} \qquad (n = 0, 2, 4, \ldots) \tag{41}$$

with an average multiplicity

$$\bar{n} = \tilde{n} \tanh \tilde{n} \tag{42}$$

and dispersion

$$D^2 = \bar{n} + \left(\frac{\tilde{n}}{\cosh \tilde{n}}\right)^2. \tag{43}$$

This is a narrow distribution and therefore leads to a negative charge-neutral correlation (Eq. 37).

Using Eq. (38) and a similar expression for the distribution $P_0(n_0)$ of neutral particles we have calculated these distributions for the case $\bar{n}_c = 11.0$, which is the average charge pion multiplicity at the highest ISR energy[1]. The results are shown in figures 3 and 4. In these figures the values of the multiplicity distributions for odd n_c and n_0 are omitted,

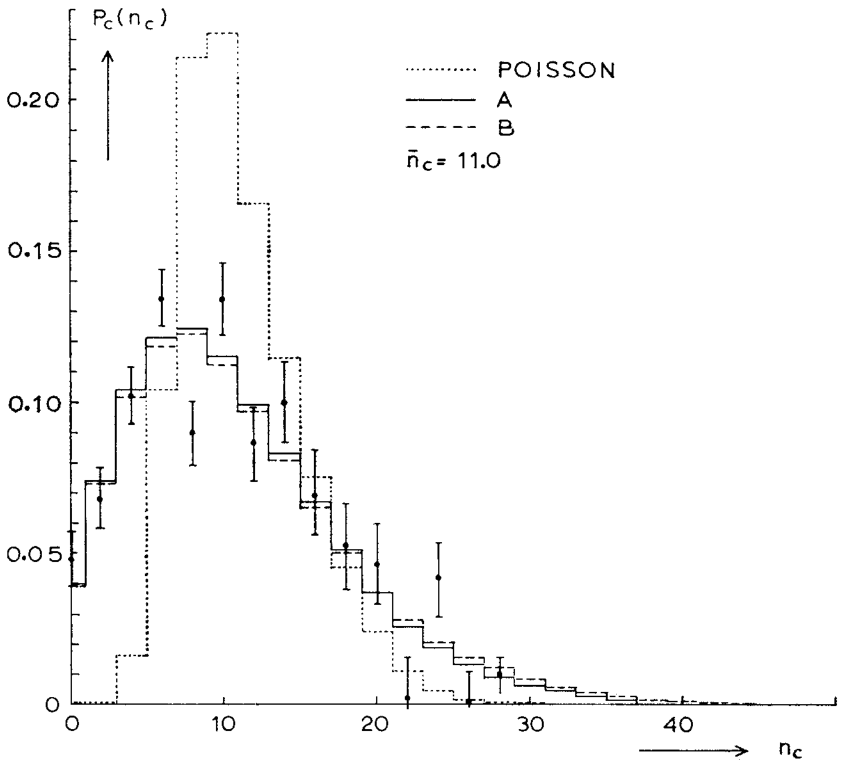

Fig. 3. Distribution of charged pion multiplicity for the average number of charged pions equal 11.0. The case A, B and Poisson are explained in the text. The data are taken from reference [5]

whereas the values in the even points are multiplied by a common factor, in order to keep normalized distributions. As long as we have not refined our method in such a way that also odd values of the total number of particles are incorporated, we will adhere to this rather artificial method.

Since we have no dynamical theory from which $P(n)$ could be calculated, we now invert the problem and consider the measured charged particle distribution $P_c(n_c)$ as input and try from this to determine the total distribution $P(n)$ and hence all other

[1] Assuming that the average number of protons is 1.7, we shift the experimental points two units to the left.

distributions and correlations. For that purpose we write the first few terms of Eq. (38)

$$
\begin{aligned}
P_c(0) &= B(0,0)P(0) = P(0), & (0)\\
P_c(1) &= B(1,0)P(2), & (1)\\
P_c(2) &= B(2,0)P(4)+B(2,1)P(2), & (2)\\
P_c(3) &= B(3,0)P(6)+B(3,1)P(4), & (3)\\
P_c(4) &= B(4,0)P(8)+B(4,1)P(6)+B(4,2)P(4), & (4)\\
P_c(5) &= B(5,0)P(10)+B(5,1)P(8)+B(5,2)P(6). & (5)
\end{aligned}
\qquad (44)
$$

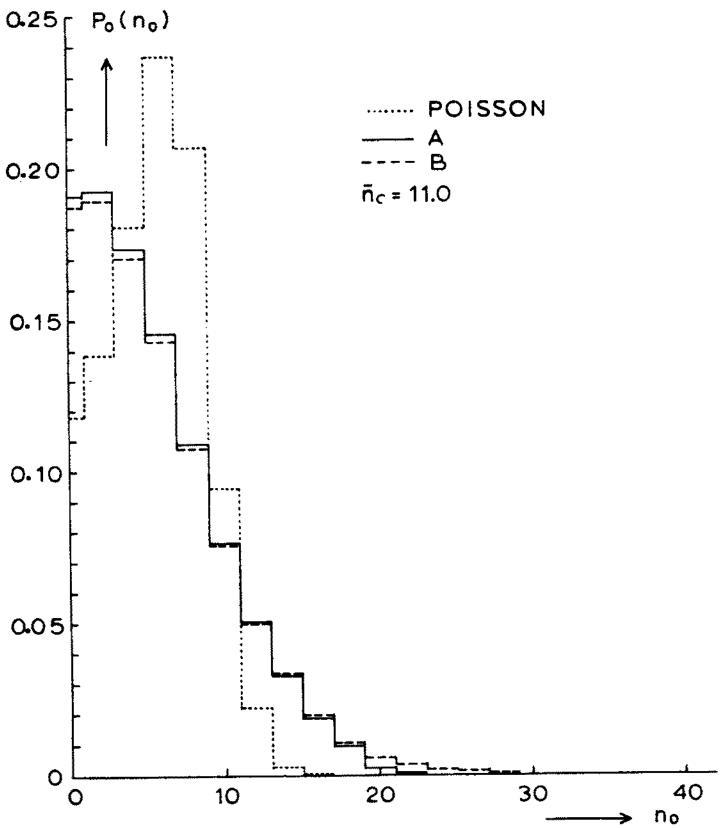

Fig. 4. Distribution of neutral pion multiplicity for the average number of charged pions equal 11.0. The cases A, B and Poisson are explained in the text

Only the $P_c(n_c)$ for even n_c are measured. We therefore define $P_c(n_c)$ for odd n_c by

$$P_c(n_c) = \tfrac{1}{2}\left(P_c(n_c-1)+P_c(n_c+1)\right) \qquad (45)$$

followed by a renormalization of the whole distribution.

Now it is clear from the above equations that, starting from the top, we can successively determine $P(0)$, $P(2)$, $P(4)$, etc. from the left-hand side. A straightforward application of this method, using the ISR data [5] for $P_c(n_c)$ as input, produced, however, a wildly fluctuating $P(n)$, which in several places even became negative. This is not surprising, since it can be shown [6] that only when the stochastic matrix $B(n_c, k)$ is a permutation

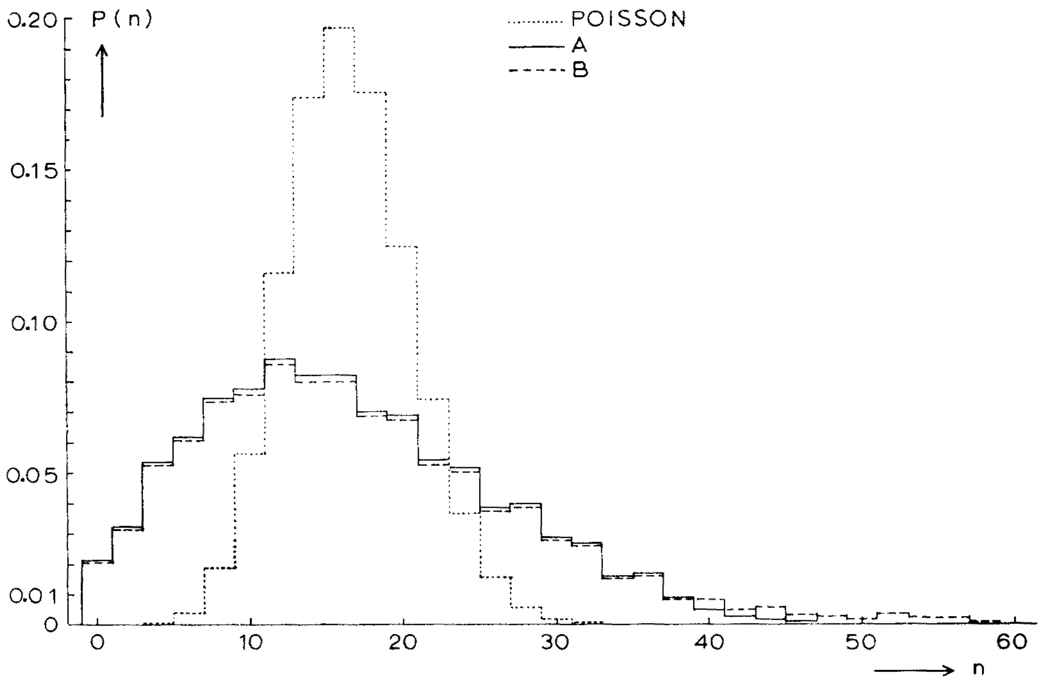

Fig. 5. Distribution of the total number of pions for the average number of charged pions equal 11.0. The cases A, B and Poisson are explained in the text

matrix, the solution $P(n)$ of equations (38) or (44) has always non-negative values. We therefore have to smooth the experimental values of $P_c(n_c)$ in order to get positive $P(n)$'s. With the $P(n)$ so determined we first calculated the $P_c(n_c)$, defined for even n_c only and again normalized. The original smoothed input $P_c(n_c)$ was reproduced within the experimental error (see figure 3). The tail of $P(n)$, however, still showed large fluctuations, due to the inaccuracy of $P_c(n_c)$ for large multiplicities. In figure 5 we show $P(n)$ for the case of $\bar{n}_c = 11.0$ in two versions: in the case A, starting from $n = 40$, the tail was replaced by a geometric progress with ratio 0.5, in the case B we did the same starting at $n = 60$. Also shown in the same figure is the Poisson distribution (41) for the same $\bar{n}_c$.

With the same $P(n)$ we then calculated the distribution of neutrals $P_0(n_0)$ (shown in figure 4) and the average number of neutral pions for a fixed number of charged ones. This quantity was defined as

$$\bar{n}_0(n_c) = \frac{\sum_{n_0} n_0 P(n_c, n_0)}{P_c(n_c)}, \tag{46}$$

where $P(n_c, n_0)$ was redefined for even n_c and n_0 only. Again for $\bar{n}_c = 11.0$ the results are shown in figure 6 together with what was obtained from the Poisson distribution (41). The different tails for the predicted values correspond to different tails of the total distribution $P(n)$, which all give charge distributions $P_c(n_c)$ in agreement with the experimental data.

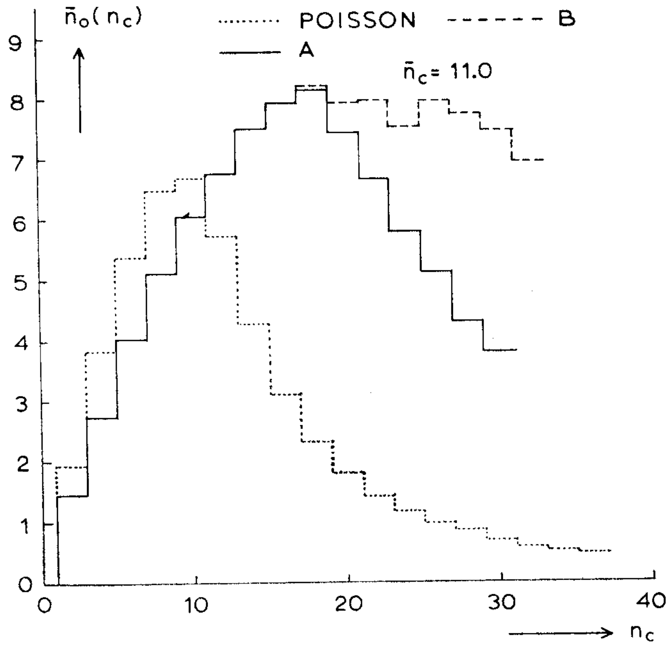

Fig. 6. The average number of neutral pions as a function of the number of charged pions for the average number of charged pions equal 11.0. The cases A, B and Poisson are explained in the text. The difference between A and B comes from the differences in the tails of $P(n)$. See text

4. Conclusions

For high energy multiparticle processes we have presented a simple picture in which the conservation of isospin was taken into account by representing a n-particle state with $I = 0$ by a closed path on a cubic lattice.

In this way we obtained for the total number of such states (defined only for even $n = 2m$)

$$g'(n) = \sum_{m_1 m_2 m_3} \binom{2m_1}{m_1}\binom{2m_2}{m_2}\binom{2m_3}{m_3} = \frac{(2m+1)!}{m!m!}, \quad (m_1+m_2+m_3 = m). \tag{47}$$

The exact number of ways in which a $I = 0$ quantum state can be constructed out of n isovector particles was calculated by Yeivin and de Shalit [7]. They found

$$g(n) = \sum_{i=0}^{n} \frac{(-1)^{n+i}}{2i+1}\binom{n}{i}\binom{2i+1}{i}. \tag{48}$$

A comparison of the two expressions is given in Table I.

TABLE I

n	0	1	2	3	4	5	6	7	8	9	72	11	12
$g(n)$	1	0	1	1	3	6	15	36	91	232	603	1585	4213
$g'(n)$	1	—	6	—	30	—	140	—	630	—	2772	—	12012

It can be shown [6] that for the exact expression the ratio $g(n+2)/g(n)$ approaches the value 9, whereas in our calculation this ratio becomes 4. Nevertheless we expect that our results as far as they can be compared with experiments are reasonable, because the idea of a random closed walk for the incorporation of isospin conservation is in principle correct.

These results can then be summed up as follows. The dynamics of the interaction determines the distribution of the total number of particles. We do not pretend to know this interaction, nor how to calculate the distribution $P(n)$. Once it is given, however, we are able, by enforcing isospin conservation, to calculate the double distribution $P(n_c, n_0)$ of the charged and of the neutral pions and hence all single particle distributions, their moments and their correlations. In particular we find two simple relations between f_{cc}, f_{00}, f_{c0} and $\bar{n}_c$ (Eqs (29) and (30)), which can be compared with experiment as soon as more data about neutral pions become available.

It has also been shown that in our model it suffices to measure the distribution of charged particles in order to find the total distribution $P(n)$ and hence all information about any multiplicity distribution. Our conjecture is that it is generally true that $P_c(n_c)$ determines $P(n)$. For this, however, it is important to have a very accurate measurement of the tail of the charge multiplicity. Especially the average number of neutral pions as a function of the number of charged pions is very sensitive to small changes in this tail, as could be seen in figure 6. The initial linear increase of this function agrees with the available low energy data.

Our main conclusion is that if our predictions about f_{00}, f_{c0} and $\bar{n}_0(n_c)$ (figures 1, 2 and 6) are verified, a strong support is found for the hypothesis that neutral distributions and correlations are nothing more than reflections of the isospin conservation.

One of the authors (W. C.) is grateful to the members of the Instituut voor Theoretische Fysica of the Rijksuniversiteit Utrecht for the warm hospitality extended to him.

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