ON TWO NUCLEON SOLITARY WAVE EXCHANGE POTENTIALS

BY M. SEBHATU* AND P. B. BURT

Department of Physics and Astronomy, Clemson University**

(Received October 18, 1977)

Using a polynomial interaction solitary wave propagator for spinless, self interacting mesons, a two nucleon potential is constructed in analogy with OBEP and previous solitary wave exchange potentials (SWEP). Since the solitary wave propagator automatically includes higher mass contributions, generalizations of OBEP are obtained without the introduction of new, arbitrary mass meson exchange. As with previous SWEP, the polynomial SWEP discussed in this paper compare favourably with phenomenological potentials but require only a few undetermined parameters.

1. *Introduction*

It is well known that phenomenological [1-3] and one-boson exchange potentials [3, 4] exist which can successfully account for N-N data up to 350 MeV. The phenomenological potentials, by their very nature, contain a large number of parameters (about 30). The OBEP have a linear field theoretical basis. Nevertheless, they regard, in most cases, masses and coupling constants of exchanged mesons as adjustable parameters and require one or more fictitious σ mesons to provide attraction at intermediate ranges. Recently [5, 6], there is a tendency to replace the σ meson contributions by multi-pion (or at least two pion) exchange mechanisms. Although this may be regarded as a step closer to the development of a "first principles" explanation of low energy nucleon-nucleon scattering, the OBEP remain at best semi-phenomenological.

The most well established part of the OBEP is the long range region which comes from one-pion exchange. Even the popular idea that vector mesons are solely responsible for the strong short range repulsion is being questioned [7]. The success of OBEP and phenomenological potentials in fitting N-N data does not, therefore, rule out other approaches based on new results in field theory. As examples of possible approaches, we have recently considered the derivation of N-N potentials from solitary wave theories of non-

* Present address: Department of Physical Sciences, Pensacola Jr. College, Pensacola, Florida 32504, USA.

** Address: Department of Physics and Astronomy, Clemson University, Clemson, South Carolina 29631, USA.

linear meson fields like the $\lambda\Phi^4$ theory and sine Gordon theory [8, 10]. These potentials are referred to as solitary wave exchange N–N potentials (SWEP).

The SWEP are obtained by employing a solitary wave propagator in lieu of the conventional linear meson field theory propagator in the usual second order perturbation theory derivation of OPEP. The SWEP, for example the $\lambda\Phi^4$ [8] and sine Gordon SWEP [10], exhibit characteristic features of successful OBEP [4] and phenomenological potentials. That is, they possess the established OPEP tail at long range ($m_\pi r \gtrsim 1.5$), can be made attractive as desired by adjusting the self interaction coupling constant(s) at intermediate ranges ($0.5 \leqslant m_\pi r \leqslant 1.5$) and at short range ($m_\pi r \leqslant 0.5$) become repulsive smoothly. These reasonable features are achieved without involving an undue number of parameters. The $\lambda\Phi^4$ and sine Gordon SWEP involve, for instance, only one parameter in addition to those associated with the standard OPEP. Physically this is accomplished because the quantized solitary waves contain "coherent operators" which have the property that the system being described is a many particle system with a mass spectrum dependent only upon the mass entering the linear theory and the type of nonlinear self interaction [11]. Thus, the potential obtained from the solitary wave propagator automatically has contributions from higher mass states, modifying the long range tail without the introduction of new parameters.

Encouraged by the $\lambda\Phi^4$ and sine Gordon SWEP results, we consider in this paper the derivation of a more general solitary wave exchange potential (polynomial SWEP) from the polynomial solitary wave field theory recently discussed by Burt [11].

The outline of this paper is as follows. In Section 2 a brief review of the quantized solitary wave theory used in this paper is given. In Section 3, expressions for the polynomial SWEP are obtained by relying on details of the derivation of the $\lambda\Phi^4$ SWEP. In Section 4 several special cases are discussed. The conclusions are given in Section 5 while some details are included in the Appendix.

2. *Review of the quantized solitary wave theory*

The solitary wave theory to be used in this paper is based on the nonlinear field equation

$$\partial_\mu\partial^\mu\Phi + m^2\Phi + \lambda_1\Phi^{2q+1} + \lambda_2\Phi^{4q+1} = 0,$$

$$\partial_\mu\partial^\mu = \partial_t^2 - \nabla^2, \qquad \hbar = c = 1, \tag{2.1}$$

where λ_1 and λ_2 are self interaction coupling constants and m is the mass of the associated linear fields. It is easy to see that for $\lambda_1 = \lambda_2 = 0$ Eq. (2.1) is the Klein Gordon equation while for $\lambda_2 = 0$, $q = 1$ or $\lambda_1 = 0$, $q = \frac{1}{2}$ the equation is that of the $\lambda\Phi^4$ theory.

A pair of exact, particular solutions (solitary wave solutions) obtained either by direct integration [11, 12] or by the method of base equations [13] are

$$\Phi_{k,q}^{(\pm)} = \varphi_k^{(\pm)}[(1-\lambda_1\varphi_k^{(\pm)2q}/4(q+1)m^2)^2 - \lambda_2\varphi_k^{(\pm)4q}/4(2q+1)m^2]^{-1/2q},$$

$$q \neq 0,\ -\tfrac{1}{2},\ -1. \tag{2.2}$$

The $\varphi_k^{(\pm)}$ in the solutions Eq. (2.2) are positive and negative frequency solutions of the Klein Gordon equation (the base equation). For the purposes of this paper we define them to be the plane wave solutions

$$\varphi_k^{(\pm)} = A_k^{(\pm)} e^{\mp ik\cdot x}/(\omega_k V)^{1/2},$$
$$k\cdot x = k_0 x_0 - \boldsymbol{k}\cdot\boldsymbol{x}, \qquad \omega_k = (\boldsymbol{k}^2+m^2)^{1/2}. \tag{2.3}$$

The $A_k^{(\pm)}$ in Eq. (2.3) are the creation and annihilation operators of the linear field theory and satisfy the commutation relations

$$[A_k^{(+)}, A_q^{(-)}] = \delta_{n_k,n_q}, \tag{2.4}$$

where in a box of volume V

$$\boldsymbol{k} = 2\pi V^{-1/3}(n_1\hat{e}_1+n_2\hat{e}_2+n_3\hat{e}_3) = 2\pi V^{-1/3} n_k. \tag{2.5}$$

The momentum space solitary wave propagator constructed from the solitary wave fields is [11]

$$P^{\text{sol}}[k, m_{qn}] = \sum_{n=0}^{\sigma} \frac{\Gamma(2qn+2)\Gamma(2qn+1)^{2qn-2}}{[m_{qn}^2+\boldsymbol{k}^2]^{qn}} \frac{[C_n^{1/2q}(\xi_q)]^2[\zeta_q V^{-q}]^{2n}}{[k^2-m_{qn}^2+i\varepsilon]}, \tag{2.6}$$

where $C_n^{1/2q}(\xi_q)$ are Gegenbauer polynomials [14], $\Gamma(a)$ are the usual gamma functions

$$m_{qn} = (2qn+1)m, \tag{2.6a}$$

$$\xi_q = \lambda_1/4\zeta_q(q+1)m^2, \tag{2.6b}$$

and

$$\zeta_q = \left[\frac{\lambda_1^2}{16(q+1)^2m^4} - \frac{\lambda_2}{4(2q+1)m^2}\right]^{1/2}. \tag{2.6c}$$

Detailed properties of the solitary wave propagator are discussed elsewhere [11]. One of the important characteristics for this paper is that the propagator has poles in k^2 at $m_{qn} = (2qn+1)\,m$ independent of the coupling constants λ_1 and λ_2. The strength of the residues at the poles depend on the coupling constants through the Gegenbauer polynomials and the factor ζ_q. In configuration space contributions from the poles (mass states) to potentials appear as superpositions of Yukawa and exponential terms with strengths that depend upon the coupling constants.

In other applications, the solitary wave propagator, Eq. (2.6), has been used in obtaining a mass formula for a sequence of spinless mesons [15] and has been used to predict oscillatory cross section for elastic baryon-antibaryon scattering [16].

3. *The N-N polynomial solitary wave exchange potential*

To obtain a SWEP we follow the standard OPEP derivation (see, for example [17]). Whenever a propagator is needed, however, we substitute a solitary wave propagator instead of the conventional linear field theory propagator. We have used this procedure earlier to obtain SWEP for the $\lambda\Phi^4$ theory and the sine Gordon theory [8, 10]. The results of this paper are obtained by using the propagator in Eq. (2.6). This generalizes the $\lambda\Phi^4$ theory and the sine Gordon theory and leads to a more general SWEP.

Following the above procedure, as for the $\lambda\Phi^4$ SWEP [8], we write the non-relativistic limit of the direct part of the lowest order solitary wave exchange N–N interaction amplitude

$$M_{\mathrm{NN}} \approx \left(\frac{g}{2M}\right)^2 (\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2)(\boldsymbol{\sigma}_1 \cdot \boldsymbol{k})(\boldsymbol{\sigma}_2 \cdot \boldsymbol{k}) \sum_{n=0}^{\infty} \left\{ \frac{\Gamma(2qn+2)(2qn+1)^{2qn-2}}{[m_{qn}^2 + k^2]^{qn+1}} \left[\frac{\zeta_q}{V^q}\right]^{2n} [C_n^{1/2q}(\xi_q)]^2 \right\}, \tag{3.1}$$

where $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$ are the usual isospin and spin Pauli matrices, g the π-N coupling constant, M the nucleon mass and $\boldsymbol{k}$ the exchanged three-momentum (the notation has been changed from that in [8] to conform with that in [5]).

In the N–N amplitude, Eq. (3.1), we deliberately neglected the exchange contribution. This is allowed without any loss of generality provided care is taken to insure that the N–N state in question is explicitly antisymmetric. This is done by choosing the sum of the total spin (S), the total isospin (T) and the total orbital angular momentum (L) of the two nucleons to be odd.

As it stands, Eq. (3.1) may be referred to as the momentum space polynomial SWEP. Momentum space potentials are useful for N–N calculations in, for example, the Lippman--Schwinger equation [5]. However, the qualitative features of the SWEP which we wish to consider are more transparent and familiar in configuration space. Therefore, we Fourier transform the non relativistic amplitude, Eq. (3.1) and obtain the polynomial SWEP as a function of position. One has

$$V_{\lambda_1,\lambda_2,q}^{(\mathrm{SWEP})}(r) = \frac{1}{(2\pi)^3}\left(\frac{g}{2n}\right)^2 [\boldsymbol{\sigma}_1 \cdot \nabla][\boldsymbol{\sigma}_2 \cdot \nabla][\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2] \sum_{n=0}^{\infty} \left\{ \Gamma(2qn+2)(2qn+1)^{2qn-2} \left[\frac{\zeta_q}{V^q}\right]^{2n} [C_n^{1/2q}(\xi_q)]^2 \int_{-\infty}^{\infty} \frac{\exp(i\boldsymbol{k}\cdot\boldsymbol{r})d^3k}{[k^2+m_{qn}^2]^{qn+1}} \right\}. \tag{3.2}$$

Evaluating the integral (see Appendix), for qn integral, Eq. (3.2) leads to

$$V_{\lambda_1,\lambda_2,q}^{(\mathrm{SWEP})}(r) = \frac{1}{2\pi}\left(\frac{g}{2M}\right)^2 [\boldsymbol{\sigma}_1 \cdot \nabla][\boldsymbol{\sigma}_2 \cdot \nabla][\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2]$$

$$\sum_{n=0}^{\infty} \left\{ \frac{(-)^{qn}(2qn+1)!(2qn+1)^{2qn-2}}{(2m_{qn})^{2qn+1}} \left[\frac{\zeta_q}{V^q}\right]^{2n} [C_n^{1/2q}(\xi_q)]^2 \right.$$

$$[L_{qn}^{(-2qn-1)}(2m_{qn}r) + 2L_{qn-1}^{(-2qn)}(2m_{qn}r)]$$

$$\left. m_{qn} \frac{\exp(-m_{qn}r)}{r} \right\}, \tag{3.3}$$

where L_a^b are Laguerre functions [20]. The polynomial SWEP has the same delta function singularity at the origin as the $\lambda\Phi^4$ SWEP or OPEP. For $n = 0$ or $\lambda_1 = \lambda_2 = 0$, Eq. (3.3) reduces to the standard OPEP, as is the case for $\lambda\Phi^4{}_1$ SWEP or sine Gordon SWEP. As expected from the field theories, the $\lambda\Phi^4$ SWEP is the $\lambda_2 = 0$, $q = 1$ or $\lambda_1 = 0$, $q = 1/2$ case of the polynomial SWEP constructed in Eq. (3.3). This and other special cases are more transparent after the potential is split into central and tensor terms [1] yielding

$$V_{\lambda_1,\lambda_2,q}^{(\mathrm{SWEP})}(m_{qn}r) = \frac{g^2}{4\pi}\left(\frac{m}{2M}\right)^2 \frac{m}{3}(\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2)$$

$$\sum_{n=0}^{\infty} \left\{ (-)^{qn}(2qn+1)! \left[\frac{\zeta_q}{(2mr)^q}\right]^{2n} [C_n^{1/2q}(\xi_q)]^2 \right.$$

$$\Big(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2[L_{qn}^{(-2qn-1)}(2m_{qn}r) + 6L_{qn-1}^{(-2qn)}(2m_{qn}r)$$

$$+ 12L_{qn-2}^{(-2qn+1)}(2m_{qn}r) + 8L_{qn-3}^{(-2qn+2)}(2m_{qn}r)]$$

$$+ S_{12}\left[\left(1 + \frac{3}{m_{qn}r} + \frac{3}{(m_{qn}r)^2}\right) L_{qn}^{(-2qn-1)}(2m_{qn}r)\right.$$

$$+ 6\left(1 + \frac{2}{m_{qn}r} + \frac{3}{(m_{qn}r)^2}\right) L_{qn-1}^{(-2qn)}(2m_{4n}r)$$

$$+ 12\left(1 + \frac{1}{m_{qn}r}\right) L_{qn-2}^{(-2qn+1)}(2m_{qn}r)$$

$$\left.\left. + 8L_{qn-3}^{(-2qn+2)}(2m_{qn}r)\right]\Big) \frac{\exp(-m_{qn}r)}{mr} \right\},$$

$$mr \neq 0,\ qn \text{ integral.} \tag{3.4}$$

---

[1] In performing the steps leading to this recombination the following identities are helpful

$$(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\nabla})(\boldsymbol{\sigma}_2 \cdot \boldsymbol{\nabla}) = \frac{1}{3}\left\{(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right] + S_{12}\left[\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right]\right\},$$

$$S_{12} = 3(\boldsymbol{\sigma}_1 \cdot \hat{r})(\boldsymbol{\sigma}_2 \cdot \hat{r}) - \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2,$$

$$\frac{d^n}{dz^n} L_\mu^{(\alpha)}(z) = (-)^n L_{\mu-n}^{(\alpha+n)}(z).$$

In Eq. (3.4) the coefficient of S_{12} is, as usual [8], the tensor term while the coefficient of $\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2$ is the central SWEP. Special cases of Eq. (3.4) will be discussed in the next section. However, it is evident from inspection of the general SWEP that it contains Yukawa and exponential contributions whose relative strengths depend upon the coupling constants through the terms in ζ_q and the Gegenbauer polynomials.

4. *Special cases of the polynomial SWEP*

Although the parameters in the polynomial SWEP must be determined by comparison with experiment, it seems worthwhile to explore some special cases obtained by making arbitrary choices for these parameters. The confrontation with experiment will be left to future studies.

As a first example, consider the case when $q = 1$. The Gegenbauer polynomials reduce to Legendre polynomials [14] and the Laguerre functions simplify to Laguerre polynomials [14]. This results in more familiar and relatively simpler expressions for the polynomial SWEP. Using standard definitions of the Legendre and generalized Laguerre polynomials, the four leading terms of the $(\lambda_1 \Phi^4 + \lambda_2 \Phi^6)$ SWEP are

$$V_{\alpha,\gamma}^{(\mathrm{SWEP})}(x) = \frac{g^2}{4\pi}\left(\frac{m}{2M}\right)^2 \frac{m}{3}(\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2)\left\{\left[\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 + S_{12}\left(1 + \frac{3}{x} + \frac{3}{x^2}\right)\right]\frac{e^{-x}}{x}\right.$$

$$+ 3\alpha^2[\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2(3x-2) + S_{12}(3x+1)]\frac{e^{-3x}}{x}$$

$$+ \frac{7 \cdot 5}{4}(2\alpha^2+\gamma)^2[\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2(5x-3) + S_{12}(5x)]e^{-5x}$$

$$\left. + \frac{7 \cdot 3 \cdot 5}{4}(2\alpha^3+\alpha\gamma)^2[\boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2(49x^2-21x-3) + S_{12}(49x^2)]e^{-7x} + \dots\right\}, \quad (x = mr) \tag{4.1}$$

where

$$\alpha = \frac{\lambda_1}{8m^3 r} \tag{4.1a}$$

and

$$\gamma = -\frac{\lambda_2}{12m^4 r}. \tag{4.1b}$$

Clearly, Eq. (4.1) reduces to the $\lambda\Phi^4$ SWEP for $\lambda_2 = 0$ [8], while for $\lambda_1 = 0$ it becomes the $\lambda\Phi^6$ SWEP (not previously discussed). In either case the leading term is the OPEP. Furthermore, the second term is not dependent upon λ_2 and thus, is identical with the second term of the $\lambda\Phi^4$ SWEP. In general, then, the higher mass states, corresponding to $n = 1$ or larger, contribute to the "interior" of the N–N potential. This is what one expects intuitively from perturbation theory, even though the SWEP is non perturbative in nature. That is, neither the fields nor the resulting propagator for the solitary waves can be obtained from a perturbation theory with the solutions of the linear Klein Gordon equation as a basis [11].

As a second example, consider the N–N system to be in a ${}^1S_0$ state. Since $S = L = 0$, the requirement $S+L+T =$ odd integer implies that $T = 1$. For this case the tensor term vanishes and $\langle \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \, \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 \rangle = -3$. The effective polynomial SWEP in Eq. (3.4) becomes

$$V^{({}^1S_0-\mathrm{SWEP})}_{\lambda_1,\lambda_2,q}(m_{qn}r) = -\frac{g^2}{4\pi}\left(\frac{m}{2M}\right)^2 m \sum_{n=0}^{\infty} (-)^{qn} \Bigg\{(2qn+1)!\left[\frac{\zeta_q}{(2mr)^q}\right]^{2n} [C_n^{1/2q}(\xi_q)]^2 [L_{qn}^{(-2qn-1)}(2m_{qn}r)+6L_{qn-1}^{(-2qn)}\, 2m_{qn}r)+12L_{qn-2}^{(-2qn+1)}(2m_{qn}r) + 8L_{qn-3}^{(-2qn+2)}(2m_{qn}r)]\,\frac{\exp(-m_{qn}r)}{mr}\Bigg\}. \tag{4.2}$$

Once again, for simplicity, we consider the special case $q = 1$, so Eq. (4.2) reduces to the $(\lambda_1 \Phi^4 + \lambda_2 \Phi^6)$ SWEP for the ${}^1S_0$ state of the two nucleons. The four leading terms are

$$V^{({}^1S_0-\mathrm{SWEP})}_{\alpha,\gamma}(x) = -\frac{g^2}{4\pi}\left(\frac{m}{2M}\right)^2 m \left[\frac{e^{-x}}{x} + 3\alpha^2(3x-2)\,\frac{e^{-3x}}{x}\;\frac{7\cdot 5}{4}\,(2\alpha^2+\gamma)^2 (5x-3)e^{-5x} + \frac{7\cdot 3\cdot 5}{4}\,(2\alpha^3+\alpha\gamma)^2(49x^2-21x-3)e^{-7x} + \dots\right]. \tag{4.3}$$

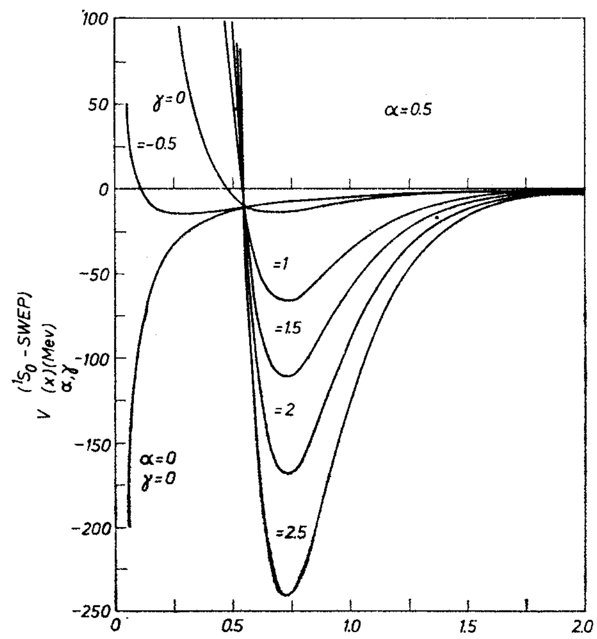

Fig. 1. An ${}^1S_0$-state polynomial $(\lambda_1\Phi^4+\lambda_2\Phi^6)$ SWEP (Eq. (4.3)). $\alpha = 0.5$ for all the curves — except for the Yukawa potential where $\alpha = \gamma = 0$. x is in pion-Compton-wavelength. The constant $g^2/4\pi(m/2M)^2 m$ in taken to be 10.5 MeV. The curves show the effect of the second (λ_2) nonlinear term with the first (λ_1) term fixed

Again, for $\lambda_2 = 0$ Eq. (4.4) is the $^1S_0$ $\lambda\Phi^4$ SWEP [8]. For $\lambda_1 = 0$ the $\lambda\Phi^6$ SWEP is

$$V_{0,\gamma}^{(^1S_0-\mathrm{SWEP})}(x) = -\frac{g^2}{4\pi}\left(\frac{m}{2M}\right)^2 m\left[\frac{e^{-x}}{x} + \tfrac{75}{4}\gamma^2(5x-3)e^{-5x} + \ldots\right]. \tag{4.4}$$

Thus, we see that the deviations from OPEP due to a $\lambda\Phi^6$ interaction contribute only exponential terms to the $^1S_0$ N–N potential. Consequently, these interactions can only modify the intermediate region of the potential since the argument of the exponential damps out the contribution relative to the OPEP potential and there is no r^{-1} contribution

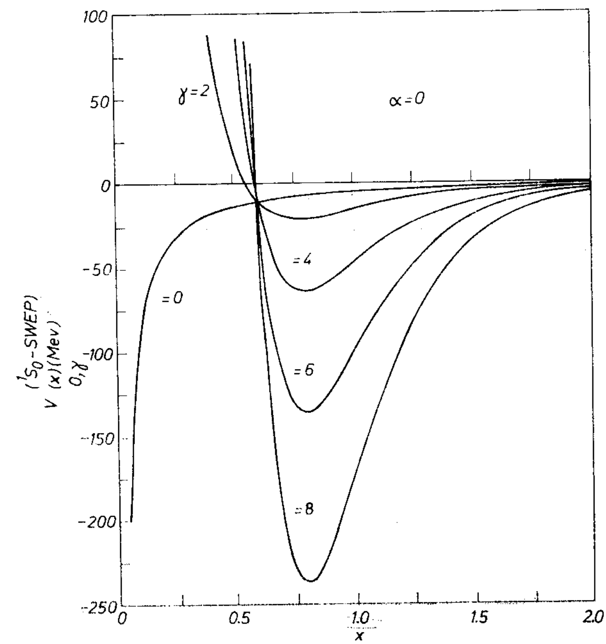

Fig. 2. An $^1S_0$-state $\lambda\Phi^6$ SWEP (Eq. (4.4)). $\alpha = 0$ for all the curves. $\gamma = 0$ is the usual O. P. E. Yukawa potential. The curves show the effect of the λ_2-term when $\lambda_1 = 0$. Notice the large values of γ used compared to those of Fig. 1

as in the OPEP. As is evident from figures 1 and 2, the $(\lambda_1\Phi^4+\lambda_2\Phi^6)$ SWEP and the $\lambda\Phi^6$ SWEP closely resemble the $\lambda\Phi^4$ and sine Gordon SWEP for the $^1S_0$ state. All approach the OPEP tail at long range ($mr \gtrsim 1.5$), they can be made as attractive as desired by adjusting λ_1 and λ_2 and at the origin they become repulsive. Note that we have set $q = 1$ in the figures. There is some indication that the radius of the repulsive core can be adjusted by varying q. Thus, the polynomial SWEP may not only be meaningfully compared with N–N scattering data, but may also be useful in interpreting nuclear matter.

5. Conclusions

As generalizations of previous SWEP [8, 10] we have derived polynomial SWEP based on the nonlinear field equation, Eq. (2.1) and its solitary wave solutions [11]. These field theories provide additional flexibility in the SWEP compared with $\lambda\Phi^4$ SWEP and sine Gordon SWEP through the dependence of the potentials on the additional coupling constant and the general exponent of the polynomial nonlinear terms. As expected, special cases of the polynomial SWEP reduce to the $\lambda\Phi^4$ SWEP. Although it is evident that the resemblance of the polynomial SWEP to phenomenological potentials is quite close, it still remains to make a detailed comparison of these theories with experiment. The SWEP exhibit features similar to the most effective phenomenological potentials (e. g., Reid soft core potentials [2]) and contain only three parameters. They are capable of adjustments in strength, especially at intermediate ranges and provide flexibility in the strength and radius of the core. These features provide motivation for using the polynomial SWEP in either N–N scattering models or nuclear matter models. This will be left for future study.

One of us (M. S.) is grateful to the University of West Florida physics faculty (in particular to Dr J. S. Marsh) for their interest in this work and use of their facilities and to Dr R. J. Philpott of Florida State University for his discussion.

APPENDIX

We consider in detail the evaluation of the integral in Eq. (3.2)

$$I = \int_{-\infty}^{\infty} \frac{\exp(i\boldsymbol{k}\cdot\boldsymbol{r})d^3k}{[k^2+m_{qn}^2]^{qn+1}}. \tag{A.1}$$

Integrating over the angle dependence, with the polar axis defined by $\boldsymbol{r}$, and using the identity

$$-i\frac{d}{dr}e^{ikr} = ke^{ikr}$$

one has

$$I = -\frac{2\pi}{r}\frac{d}{dr}\int_{-\infty}^{\infty}\frac{\exp(ikr)}{[k^2+m_{qn}^2]^{qn+1}}. \tag{A.2}$$

This integral is [18]

$$I = -\frac{2\pi}{r}\frac{d}{dr}\left[\frac{2\pi}{(2m_{qn})^{qn+1}}\frac{r^{qn}}{\Gamma(qn+1)}W_{0,-qn-\frac{1}{2}}(2m_{qn}r)\right], \tag{A.3}$$

where $W_{a,b}$ are Whittaker functions [19]. These may be expressed in terms of generalized Laguerre functions [20]. For qn integral,

$$W_{0,-qn-\frac{1}{2}}(2m_{qn}r) = (-)^{qn}\Gamma(qn+1)(2m_{qn}r)^{-qn}$$
$$\exp(-m_{qn}r)L_{qn}^{(-2qn-1)}(2m_{qn}r). \tag{A.4}$$

Using the relation

$$\frac{d}{dr} L_{qn}^{(-2qn-1)}(2m_{qn}r) = -2m_{qn}L_{qn-1}^{(-2qn)}(2m_{qn}r) \tag{A.5}$$

along with Eq. (A.4) in Eq. (A.3), one has finally

$$I = (-)^{qn} \frac{4\pi^2}{(2m_{qn})^{2qn+1}} (m_{qn}) \frac{\exp(-m_{qn}r)}{r}$$
$$[L_{qn}^{(-2qn-1)}(2m_{qn}r) + 2L_{qn-1}^{(-2qn)}(2m_{qn}r)]. \tag{A.6}$$

This is the result used in the text.

REFERENCES

[1] T. Hamada, I. D. Johnston, *Nucl. Phys.* **34**, 382 (1962).
[2] R. V. Reid, *Ann. Phys.* **50**, 411 (1968).
[3] M. J. Moravcsik, *Rep. Prog. Phys.* **35**, 587 (1972).
[4] K. Erkelenz, *Phys. Lett.* **13C**, 191 (1974).
[5] G. E. Brown, A. D. Jackson, *The Nucleon-Nucleon Interaction*, New York 1976.
[6] K. Holinde, R. Machleidt, *Nucl. Phys.* **A28**, 429 (1977).
[7] H. G. Kruse, H. G. Dosch, Heidelberg Preprint (1976).
[8] M. Sebhatu, *Nuovo Cimento* **33A**, 568 (1976).
[9] P. B. Burt, M. Sebhatu, *Acta Phys. Pol.* **B7**, 729 (1976).
[10] M. Sebhatu, *Lett. Nuovo Cimento* **16**, 463 (1976).
[11] P. B. Burt, *Acta Phys. Pol.* **B7**, 617 (1976); *Nonperturbative Self Interactions*, in preparation.
[12] P. B. Burt, *Phys. Rev. Lett.* **32**, 1080 (1974).
[13] P. B. Burt, J. L. Reid, *J. Math. Anal. Appl.* **55**, 43 (1976).
[14] Bateman Manuscript Project, A. Erdelyi, editor, *Higher Transcendental Functions*, V. 2, New York 1953.
[15] P. B. Burt, *Lett. Nuovo Cimento* **13**, 26 (1975).
[16] P. B. Burt, M. Sebhatu, *Lett. Nuovo Cimento* **13**, 104 (1975).
[17] J. D. Bjorken, S. D. Drell, *Relativistic Quantum Mechanics*, New York 1964.
[18] Bateman Manuscript Project, A. Erdelyi, editor, *Tables of Integral Transforms*, V. 1, New York 1954.
[19] Bateman Manuscript Project, A. Erdelyi, editor, *Higher Transcendental Functions*, V. 1, New York 1953.
[20] H. Buchholz, *The Confluent Hypergeometric Function*, New York 1969.