Vol. B9 (1978) ACTA PHYSICA POLONICA No 6

SOME EXACT SOLUTIONS OF CHARGED FLUID SPHERES IN GENERAL RELATIVITY

BY T. SINGH AND R. B. S. YADAV

Applied Mathematics Section, Institute of Technology, Banaras Hindu University, Varanasi*

(Received October 27, 1977)

The paper presents a family of interior solutions of the Einstein-Maxwell field equations of general relativity for a static, spherically symmetric distribution of charged fluid. The family of solutions have been matched with the Reissner-Nordstrom metric at the boundary. The solution of Adler can be obtained from these when charge is absent.

1. Introduction

The Einstein–Maxwell field equations in the presence of matter and charge form a highly non-linear system of equations. So a small number of exact solutions have been obtained. Bonnor (1960), Efinger (1965), Kyle and Martin (1967), Krori and Barua (1975) and Nduka (1976, 1977) have obtained internal solutions for static spherically symmetric charged fluid spheres under different conditions.

For the charged static spherically symmetric fluid of density ϱ, pressure p and total charge Q, the field equations reduce to three coupled ordinary differential equations involving these fluid variables and two metric functions. In order to solve this system, it is necessary to specify in some manner two of the unknowns or to introduce subsidiary relations between them i.e. specify an equation of state. For example Nduka (1977) has solved these by taking $e^{-\lambda}$ = constant and a suitable form of total charge Q.

In the present paper we have obtained some exact solutions for the spherically symmetric charged fluid distribution using an entirely different technique, namely a specific choice of metric function $\nu(r)$ and charge $Q(r)$. We have also shown that they may reduce to Adler's (1974) case in the absence of charge.

* Address: Applied Mathematics Section, Institute of Technology, Banaras Hindu University, Varanasi 221005, India.

2. *The field equations and their solutions*

We use here the spherically symmetric line element

$$ds^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - r^2(d\Theta^2 + \sin^2\Theta d\Phi^2), \tag{2.1}$$

where λ and ν are functions of r only.

The Einstein–Maxwell equations for the charged perfect fluid distribution in general relativity are

$$G_{ij} = -8\pi T_{ij}, \tag{2.2}$$

$$[(-g)^{1/2}F^{ij}]_{,j} = 4\pi J^i(-g)^{1/2}, \tag{2.3}$$

$$F_{[ij,k]} = 0, \tag{2.4}$$

where T_{ij} is the energy momentum tensor, J^i is the current four vector and $G_{ij} = R_{ij} - \frac{1}{2}Rg_{ij}$ is the Einstein tensor with R_{ij} as the Ricci tensor and R the scalar of curvature tensor.

For the system under study the energy momentum tensor T^i_j has two parts viz. t^i_j and E^i_j for matter and charges respectively

$$T^i_j = t^i_j + E^i_j, \tag{2.5}$$

where

$$t^i_j = [(\varrho + p)u^i u_j - \delta^i_j p] \tag{2.6}$$

with $u^j u_j = 1$. The nonvanishing components of t^i_j are

$$t^4_4 = \varrho, \quad t^1_1 = t^2_2 = t^3_3 = -p.$$

The electromagnetic energy tensor E^i_j in terms of field tensor F_{ij} is given by

$$E^i_j = -F_{jk}F^{ik} + \frac{1}{4}\delta^i_j F_{\mu\nu}F^{\mu\nu}. \tag{2.7}$$

Due to spherical symmetry, the only non-vanishing components of field tensor F^{ij} are $F^{41} = -F^{14}$. It then follows that the nonzero components of E^i_j are

$$E^4_4 = E^1_1 = -E^2_2 = -E^3_3 = -\frac{1}{8\pi} g_{44}g_{11}(F^{41})^2.$$

By this choice of field tensor F^{ij} equation (2.4) is clearly satisfied.

Now we get from equation (2.3)

$$F^{41} = \frac{Q(r)e^{-N}}{r^2}, \tag{2.8}$$

where $N = (\lambda + \nu)/2$ and $Q(r)$ represents the total charge within a sphere of radius r, viz.,

$$Q(r) = 4\pi \int_0^r J^4 r^2 e^N dr. \tag{2.9}$$

From equation (2.9) we see that outside the fluid sphere $Q(r)$ is a constant Q_0 which is the total charge. Then from (2.8) one finds the asymptotic form of the electric field as Q_0/r^2.

Thus the field equations may be written as (Adler et al., 1974),

$$8\pi\varrho + 8\pi E_4^4 = e^{-\lambda}\left[\frac{\lambda'}{r} - \frac{1}{r^2}\right] + \frac{1}{r^2}, \tag{2.10}$$

$$8\pi p - 8\pi E_1^1 = e^{-\lambda}\left[\frac{v'}{r} + \frac{1}{r^2}\right] - \frac{1}{r^2}, \tag{2.11}$$

$$8\pi p - 8\pi E_2^2 = e^{-\lambda}\left[\frac{v''}{2} + \frac{(v')^2}{4} - \frac{\lambda' v'}{4} + \frac{v' - \lambda'}{2r}\right]. \tag{2.12}$$

Eliminating p from (2.11) and (2.12) we get

$$v'' + \frac{(v')^2}{2} - \frac{1}{r}(v' + \lambda') - \frac{\lambda' v'}{2} + \frac{1}{2r^2}(2 - 32\pi E_1^1 r^2)e^{\lambda} - \frac{2}{r^2} = 0. \tag{2.13}$$

In equation (2.13) we can find one of functions v and λ if the other is given. For this we choose v of the form

$$v = 2\log y. \tag{2.14}$$

Using equations (2.8) and (2.14) in (2.13) we get the second order differential equation

$$y'' - \left(\frac{1}{r} + \frac{\lambda'}{r}\right)y' + \left(\frac{e^{\lambda}}{r^2} - \frac{\lambda'}{2r} - \frac{1}{r^2} - \frac{2Q^2(r)e^{\lambda}}{r^4}\right)y = 0, \tag{2.15}$$

which is generalization of Wyman's equation (Wyman, 1949).

Let us assume that total charge Q is given by

$$Q = Ar^n, \tag{2.16}$$

where A is a proportionality constant and n is a positive integer.

Equations (2.15) and (2.16) together give

$$y'' - \left(\frac{1}{r} + \frac{\lambda'}{r}\right)y' + \left(\frac{e^{\lambda}}{r^2} - \frac{\lambda'}{r} - \frac{1}{r^2} - 2A^2 r^{2n-4} e^{\lambda}\right)y = 0. \tag{2.17}$$

Now we define

$$e^{-\lambda} = \tau(r). \tag{2.18}$$

Then equation (2.17) may be written as a first order differential equation in $\tau(r)$ viz.

$$\tau' - \tau\left[\frac{2(y + ry' - r^2 y'')}{r(y + ry')}\right] = \frac{-2y(1 - 2A^2 r^{2n-2})}{r(y + ry')}. \tag{2.19}$$

This has the solution

$$\tau(r) = \exp\left[-F(r)\right]\left\{\int^r \exp\left[F(r)\right]g(r)dr + C\right\}, \tag{2.20}$$

where

$$f = \frac{-2(y+ry'-r^2y'')}{r(y+ry')},$$

$$g = \frac{-2y(1-2A^2r^{2n-2})}{r(y+ry')}, \qquad F(r) = \int^r f(r)dr$$

and C is a constant of integration to be fixed by the boundary conditions.

3. *Model solution*

We want to find a solution of equation (2.17). Wilson (1969), Nduka (1976), Krori and Barua (1975) have solved this by imposing various conditions that simplify the equation and allow immediate integration. Once ν and λ are obtained, ϱ and p follow directly from equations (2.10) and (2.11). Equation (2.20) together with (2.8), (2.10), (2.11) and (2.16) represents all solutions for static spherically symmetric charged fluid bodies. As a matter of fact, one should not hope that all solutions will be physically reasonable. Only a subclass of these solutions, corresponding to certain functions $\nu(r)$ will be physically reasonable. Hence a specific choice of $\nu(r)$ is essential. Here we specify $\nu(r)$ in such a manner that equation (2.20) can be immediately integrated. Such a choice is that y satisfies the equation

$$r^2y''-ry'-2A^2r^{2n-2}y = 0. \tag{3.1}$$

In this paper we consider $n = 1$ which reduces equation (3.1) to

$$r^2y''-ry'-2A^2y = 0. \tag{3.2}$$

On letting $p = -1$ and $q = -2A^2$, (3.2) is transformed into well known Euler's equation

$$r^2y''+pry'+qy = 0. \tag{3.3}$$

The indicial equation for (3.3) is

$$m^2+(p-1)m+q = 0. \tag{3.4}$$

Three different cases arise, depending on the value of discriminant (Ritger and Rose, 1968)

$$\Delta = (p-1)^2-4q. \tag{3.5}$$

Case 1. $\Delta > 0$. This requires $A^2 > -\frac{1}{2}$. The solution is

$$y = a_1r^l+b_1r^m, \tag{3.6}$$

where a_1 and b_1 are constants to be fixed by boundary conditions and

$$l = 1+\eta, \; m = 1-\eta \quad \text{for} \quad \eta = \sqrt{1+2A^2}. \tag{3.7}$$

When (3.6) is used in conjuction with equation (2.20) $\tau(r)$ can be readily obtained as

$$\tau(r) = (2-\eta^2)^{-1} + c_1 r^{2(2-\eta^2)/(2-\eta)}[a_1(2+\eta)r^{2\eta} + b_1(2-\eta)]^{-2(2-\eta^2)/(4-\eta^2)}, \tag{3.8}$$

where c_1 is a constant of integration to be fixed by the boundary conditions.

Now y and τ are known i.e. ν and λ are known. The electromagnetic energy tensor is given by

$$8\pi r^2 E_1^1 = 8\pi r^2 E_4^4 = -8\pi r^2 E_2^2 = A^2. \tag{3.9}$$

Hence from equations (2.10) and (2.11), density ϱ and pressure p are given by

$$8\pi r^2 \varrho(r) = 1-\tau(r) - 2\{(2-\eta^2)\tau(r)-1\}\,(a_1 r^{2\eta}+b_1)$$
$$\{a_1(2+\eta)r^{2\eta}+b_1(2-\eta)\}^{-1} - A^2, \tag{3.10}$$

$$8\pi r^2 p(r) = \tau(r)\,[a_1(3+2\eta)r^{2\eta}+b_1(3-2\eta)]\,(a_1 r^{2\eta}+b_1)^{-1} - 1 + A^2. \tag{3.11}$$

We now impose the following boundary conditions: (1) The function $e^{-\lambda}$ i.e. $\tau(r)$ is continuous across the boundary ($r \equiv r_0$) of the fluid sphere. (2) The function e^{ν} is continuous across the boundary ($r \equiv r_0$) of the fluid sphere. (3) The function $e^{\nu}\nu'$ is continuous across the boundary ($r \equiv r_0$) of the fluid sphere.

The line element for $r > r_0$ is given by the Reissner–Nordstrom metric

$$ds^2 = \left(1 - \frac{2M}{r} - \frac{Q_0^2}{r^2}\right)dt^2 - \left(1 - \frac{2M}{r} - \frac{Q_0^2}{r^2}\right)^{-1} dr^2$$
$$- r^2(d\Theta^2 + \sin^2\Theta d\Phi^2), \tag{3.12}$$

where $Q_0 = Q(r_0)$ and M is the total mass of the sphere given by

$$M = 4\pi \int_0^{r_0} \varrho(r) r^2 dr. \tag{3.13}$$

Using the above boundary conditions, the three constants of integrations a_1, b_1 and c_1 are given by

$$a_1 = (1-\mathscr{C}-m\mathscr{C}-Q_0^2/r_0^2)\beta^{-1} r_0^{-l}, \tag{3.14}$$

$$b_1 = -\left(1-\mathscr{C}-l\mathscr{C}-\frac{Q_0^2}{r_0^2}\right)\beta^{-1} r_0^{-m}, \tag{3.15}$$

$$c_1 = \{\mathscr{C}-(2-\eta^2)^{-1}\}^{-2(2-\eta^2)/(2-\eta)}$$
$$[a_1(2+\eta)r_0^{2\eta}+b_1(2-\eta)]^{2(2-\eta^2)/(4-\eta^2)}, \tag{3.16}$$

where

$$\beta = l-m$$

and

$$\mathscr{C} = 1 - \frac{2M}{r_0} + \frac{Q_0^2}{r_0^2}.$$

Case 2. $\Delta = 0$ i.e. $A = \pm i(2)^{-1/2}$. The solution in this case is

$$y = r(a_2 \log r + b_2), \tag{3.17}$$

$$e^{-\lambda} = \tau(r) = 1 + c_2 e^{2b_2/a_2}(2a_2 \log r + 2b_2 + a_2)^{-1} r^2, \tag{3.18}$$

where a_2, b_2 and c_2 are constants to be fixed by boundary conditions. Density and pressure in this case are given by

$$8\pi r^2 \varrho(r) = 1 - \tau(r)\{3 - 2a_2(2a_2 \log r + 2b_2 + a_2)^{-1}\} - A^2, \tag{3.19}$$

$$8\pi r^2 p(r) = \tau(r)\{3 + 2a_2(a_2 \log r + b_2)^{-1}\} - 1 + A^2, \tag{3.20}$$

where

$$a_2 = \frac{1}{2r_0\sqrt{\mathscr{C}}}\left(1 - 3\mathscr{C} - \frac{Q_0^2}{r_0^2}\right), \tag{3.21}$$

$$b_2 = \frac{\sqrt{\mathscr{C}}}{r_0} - \frac{\log r_0}{2r_0\sqrt{\mathscr{C}}}\left(1 - 3\mathscr{C} - \frac{Q_0^2}{r_0^2}\right), \tag{3.22}$$

$$c_2 = (\mathscr{C} - 1)\exp\left(-\frac{2b_2}{a_2}\right)\left(\frac{2\sqrt{\mathscr{C}}}{r_0} + a_2\right) r_0^{-2}, \tag{3.23}$$

where $\mathscr{C}$ has the same meaning as in case 1.

Case 3. $\Delta < 0$ i.e. $A < \pm i(2)^{-1/2}$. The roots of the indicial equation (3.4) are conjugate complex numbers

$$m_1 = \sigma + i\delta \quad \text{and} \quad m_2 = \sigma - i\delta,$$

where σ and δ will depend on the value of A. The solution in this case is

$$y = r^{\sigma}(\psi_1 r^{i\delta} + \psi_2 r^{-i\delta}), \tag{3.24}$$

where ψ_1 and ψ_2 are constants. This solution can be expressed in terms of real functions by noting that

$$r^{i\delta} = e^{i\delta \log r}$$

$$= \cos(\delta \log r) + i \sin(\delta \log r). \tag{3.25}$$

Hence the solution (3.24) can be written as

$$y = r^{\sigma}[a_3 \cos x + b_3 \sin x], \tag{3.26}$$

and

$$e^{-\lambda} = \tau(r) = 1 + c_3 e^{-L + \frac{\omega_3 \pi}{2}} \exp\left\{-\left(\frac{\omega_1}{\omega_1^2+\omega_2^2} + \frac{\omega_3}{\omega_3^2+\omega_4^2}\right)\right\} x$$

$$(\omega_1 \cos x + \omega_2 \sin x)^{-\omega_2/(\omega_1{}^2+\omega_2{}^2)}$$

$$(\omega_3 \sin x + \omega_4 \cos x)^{\omega_4/(\omega_3{}^2+\omega_4{}^2)}, \tag{3.27}$$

where

$$\omega_1 = 2 + \frac{b_3 \delta}{a_3}, \quad \omega_2 = \frac{2b_3}{a_3} - \delta,$$

$$\omega_3 = 2 - \frac{a_3 \delta}{b_3}, \quad \omega_4 = \frac{2a_3}{b_3} + \delta,$$

$$x = \delta \log r \tag{3.28}$$

and

$$L = -\frac{2(1-2A^2)}{b_3}.$$

From equations (2.10) and (2.11) density and pressure are given by

$$8\pi r^2 \varrho(r) = (\tau(r)-1)\left[-1 + r\left\{-\left(\frac{\omega_1}{\omega_1^2+\omega_2^2} + \frac{\omega_3}{\omega_3^2+\omega_4^2}\right)\right.\right.$$

$$-\left(\frac{\omega_2}{\omega_1^2+\omega_2^2}\right)(\omega_1 \cos x + \omega_2 \sin x)^{-1}$$

$$(-\omega_1 \sin x + \omega_2 \cos x) + \frac{\omega_4}{\omega_3^2+\omega_4^2}(\omega_3 \sin x + \omega_4 \cos x)^{-1}$$

$$\left.\left.(\omega_3 \cos x - \omega_4 \sin x)\right\}\right] - A^2, \tag{3.29}$$

$$8\pi r^2 p(r) = \tau + \tau[r \log(2\sigma) + 2(a_3 \cos x + b_3 \sin x)^{-1}$$

$$(-a_3 \delta \sin x + b_3 \delta \cos x)] - 1 + A^2, \tag{3.30}$$

where x, ω_1, ω_2, ω_3, ω_4 have the same meaning as in (3.28). The constants a_3, b_3 and c_3 are fixed by boundary condition and have the value

$$a_3 = \cos x_0 r_0^{-\sigma} \sqrt{\mathscr{C}} - \frac{\sin x_0}{2r_0^{\sigma+1}\sqrt{\mathscr{C}}}(1 - 2\mathscr{C} - 2\sigma\mathscr{C} - Q_0^2/r_0^2), \tag{3.31}$$

$$b_3 = \sin x_0 r_0^{-\sigma} \sqrt{\mathscr{C}} + \frac{\cos x_0}{2r_0^{\sigma+1}\sqrt{\mathscr{C}}}(1 - 2\mathscr{C} - 2\sigma\mathscr{C} - Q_0^2/r_0^2), \tag{3.32}$$

$$c_3 = (\mathscr{C} - 1)\exp\left(L + \frac{\omega_3 \pi}{2}\right)\exp\left\{\frac{\omega_1}{\omega_1^2+\omega_2^2} + \frac{\omega_3}{\omega_3^2+\omega_4^2}\right\} x_0$$

$$(\omega_1 \cos x_0+\omega_2 \sin x_0)^{\omega_2/(\omega_1^2+\omega_2^2)}$$

$$(\omega_3 \sin x_0+\omega_4 \cos x_0)^{-\omega_4/(\omega_3^2+\omega_4^2)}, \tag{3.33}$$

where $\mathscr{C}$ has the same meaning as in cases 1 and 2, ω_1, ω_2, ω_3, ω_4, L are given by equation (3.28) and

$$x_0 = \delta \log r_0.$$

In addition the conditions $p \geqslant 0$ and $\varrho \geqslant 0$ in the interior of the fluid sphere will impose further conditions on these solutions. We therefore restrict our solutions to those values of the constants for which the pressure and density are positive.

If in equations (3.6)—(3.16) we set $A = 0$ the results coincide with those of Adler (1974). Thus our solutions may be considered as generalizations of those obtained by Adler.

REFERENCES

Adler, R. J., *J. Math. Phys.* **15**, 727 (1974).

Adler. R. J., Bazin, M., Schiffer, M. M., *Introduction to General Relativity*, McGraw-Hill, New York 1974, Chapter 14.

Bonnor, W. B., *Z. Phys.* **160**, 59 (1960).

Efinger, H. J., *Z. Phys.* **188**, 31 (1965).

Krori, K. D., Barua, J., *J. Phys.* **A8**, 508 (1975).

Kyle, C. F., Martin, A. W., *Nuovo Cimento* **50**, 583 (1967).

Nduka, A., *Gen. Relativ. Gravitation* **7**, 501 (1976).

Nduka, A., *Acta Phys. Pol.* **B8**, 75 (1977).

Ritger, P. D., Rose, N. J., *Differential Equatio* [illegible] pp. 144–146.

Wilson, S. J., *Can. J. Phys.* **47**, 2401 (1969).

Wyman, M., *Phys. Rev.* **75**, 1930 (1949).