RELATIVISTIC BARRIERS FOR TWO SPIN-1/2 PARTICLES

By W. Królikowski

Institute of Theoretical Physics, Warsaw University*

and J. Rzewuski

Institute of Theoretical Physics, Wrocław University**

(*Received January 9, 1978*)

We separate the angular coordinates in relativistic Breit equation with a central potential, and then split it into a system of 16 radial equations. We show in the case of positive potentials $V = \alpha/r$ and $V = \mu^2 r$ that after elimination of all wave-function components but one some relativistic barriers appear between or around two Dirac particles. An energy-dependent radius of such a barrier separates the relative space of two particles into an inner region of partial confinement and an outer region of repulsion. The existence and character of solutions in the partial confinement regions require further investigation.

1. Introduction

Recently, the discovery of the narrow resonances in e^+e^- annihilation raised much interest [1, 2] in the binding problem of two spin-1/2 particles and the role of relativistic effects there. The most popular approach to this problem was based on the non-relativistic Pauli approximation to the relativistic Breit equation,

$$[E-(\vec{\alpha}^{(1)}\cdot\vec{p}+\beta^{(1)}m^{(1)})-(-\vec{\alpha}^{(2)}\cdot\vec{p}+\beta^{(2)}m^{(2)})-V(\vec{r})]\psi(\vec{r}) = 0, \tag{1}$$

where the phenomenological potential $V(\vec{r})$ was usually taken in the form

$$V(\vec{r}) = -\frac{\alpha_s}{r}+\mu^2 r+V_0+\text{Breit-like terms} \tag{2}$$

suggested partly by the quantum chromodynamics and partly by the quark confinement.

The Breit equation (1) represents the one-time formulation of the relativistic two-body problem which has been derived several years ago from the two-time formulation provided

---

* Address: Instytut Fizyki Teoretycznej, Uniwersytet Warszawski, Hoża 69, 00-681 Warszawa, Poland.

** Address: Instytut Fizyki Teoretycznej, Uniwersytet Wrocławski, Cybulskiego 36, 50-205 Wrocław, Poland.

by the Bethe-Salpeter equation [3] and also from the quantum field theory [4]. At the same time a method has been described, how to separate the angular coordinates from the one--time relativistic wave equation for two spin-1/2 particles with a general interaction energy [5].

This method, when applied to the Breit equation (1) with a potential of the form

$$V(\vec{r}) = V(r)-\tfrac{1}{2}\left[\vec{\alpha}^{(1)}\cdot\vec{\alpha}^{(2)}+\frac{(\vec{r}\cdot\vec{\alpha}^{(1)})(\vec{r}\cdot\vec{\alpha}^{(2)})}{r^2}\right]V'(r), \tag{3}$$

gives the following radial equation [6]:

$$\begin{aligned}\Big\{E+i(\alpha_3^{(1)}-\alpha_3^{(2)})\left[\frac{d}{dr}+\frac{1+\frac{1}{2}(\alpha_1^{(1)}\alpha_1^{(2)}+\alpha_2^{(1)}\alpha_2^{(2)})}{r}\right]\\ -i(\alpha_1^{(1)}-\alpha_1^{(2)})\frac{\alpha_2^{(1)}\alpha_2^{(2)}\sqrt{j(j+1)}}{r}-\beta^{(1)}m^{(1)}-\beta^{(2)}m^{(2)}\\ -V(r)+\tfrac{1}{2}(\alpha_1^{(1)}\alpha_1^{(2)}+\alpha_2^{(1)}\alpha_2^{(2)}+2\alpha_3^{(2)}\alpha_3^{(2)})V'(r)\Big\}\psi(r)=0,\end{aligned} \tag{4}$$

where j denotes the total angular momentum. Here $V(r)$ and $V'(r)$ are arbitrary radial potentials. In the case of electromagnetic interaction we have $V(r) = V'(r) = \pm\alpha/r$. In the static approximation we put $V(\vec{r}) = V(r)$.

2. *System of radial equations*

In a given representation of Dirac matrices, the radial equation (4) can be written as a system of 16 equations. For instance, let us use the particular representation defined by formulae

$$\begin{aligned}\alpha_1^{(1)} &= \sigma_3\times\sigma_1\times\mathbf{1}\times\mathbf{1}, & \alpha_1^{(2)} &= \mathbf{1}\times\sigma_1\times\sigma_3\times\mathbf{1},\\ \alpha_2^{(1)} &= \sigma_1\times\mathbf{1}\times\mathbf{1}\times\sigma_1, & \alpha_2^{(2)} &= \sigma_1\times\sigma_3\times\mathbf{1}\times\sigma_1,\\ \alpha_3^{(1)} &= \sigma_2\times\sigma_1\times\mathbf{1}\times\mathbf{1}, & \alpha_3^{(2)} &= \mathbf{1}\times\sigma_1\times\sigma_2\times\mathbf{1},\\ \beta^{(1)} &= \sigma_1\times\sigma_1\times\mathbf{1}\times\sigma_3, & \beta^{(2)} &= \mathbf{1}\times\sigma_1\times\sigma_1\times\mathbf{1},\end{aligned} \tag{5}$$

where σ_1, σ_2, σ_3 and $\mathbf{1}$ are the usual 2×2 Pauli matrices. Then we obtain the system of 16 equations listed in the static case in Table I [6], where the components f and g are defined by relations

$$\begin{aligned}f_1^- &= \frac{\psi_1-\psi_6}{\sqrt{2}}, & f_2^- &= \frac{\psi_4-\psi_7}{\sqrt{2}}, & f_3^+ &= \frac{\psi_1+\psi_6}{\sqrt{2}}, & f_4^+ &= \frac{\psi_4+\psi_7}{\sqrt{2}},\\ g_1^- &= \frac{\psi_3-\psi_8}{\sqrt{2}}, & g_2^- &= \frac{\psi_2-\psi_5}{\sqrt{2}}, & g_3^+ &= \frac{\psi_3+\psi_8}{\sqrt{2}}, & g_4^+ &= \frac{\psi_2+\psi_5}{\sqrt{2}}\end{aligned} \tag{6}$$

TABLE I

The radial equation (4) in the static case written in the representation (5)

$$\frac{d}{dr} f_2^{\pm} \pm \frac{m^{(1)} \mp m^{(2)}}{2} f_4^{\mp} + \frac{E-V}{2} f_3^{\mp} = 0$$

$$\pm \frac{m^{(1)} \mp m^{(2)}}{2} f_3^{\mp} + \frac{E-V}{2} f_4^{\mp} + \frac{i\sqrt{j(j+1)}}{r} g_2^{\pm} = 0$$

$$\pm \frac{m^{(1)} \pm m^{(2)}}{2} f_2^{\pm} + \frac{E-V}{2} f_1^{\pm} = 0$$

$$-\left(\frac{d}{dr} + \frac{2}{r}\right) f_3^{\mp} \pm \frac{m^{(1)} \pm m^{(2)}}{2} f_1^{\pm} + \frac{E-V}{2} f_2^{\pm} + \frac{i\sqrt{j(j+1)}}{r} g_4^{\mp} = 0$$

$$\left(\frac{d}{dr} + \frac{1}{r}\right) g_2^{\pm} \pm \frac{m^{(1)} \mp m^{(2)}}{2} g_4^{\mp} + \frac{E-V}{2} g_3^{\mp} = 0$$

$$\pm \frac{m^{(1)} \mp m^{(2)}}{2} g_3^{\mp} + \frac{E-V}{2} g_4^{\mp} - \frac{i\sqrt{j(j+1)}}{r} f_2^{\pm} = 0$$

$$\pm \frac{m^{(1)} \pm m^{(2)}}{2} g_2^{\pm} + \frac{E-V}{2} g_1^{\pm} = 0$$

$$-\left(\frac{d}{dr} + \frac{1}{r}\right) g_3^{\mp} \pm \frac{m^{(1)} \pm m^{(2)}}{2} g_1^{\pm} + \frac{E-V}{2} g_2^{\pm} - \frac{i\sqrt{j(j+1)}}{r} f_4^{\mp} = 0$$

and

$$f_1^+ = \frac{\psi_9 - \psi_{14}}{\sqrt{2}}, \quad f_2^+ = \frac{\psi_{12} - \psi_{15}}{\sqrt{2}}, \quad f_3^- = \frac{\psi_9 + \psi_{14}}{\sqrt{2}}, \quad f_4^- = \frac{\psi_{12} + \psi_{15}}{\sqrt{2}},$$

$$g_1^+ = \frac{\psi_{11} - \psi_{16}}{\sqrt{2}}, \quad g_2^+ = \frac{\psi_{10} - \psi_{13}}{\sqrt{2}}, \quad g_3^- = \frac{\psi_{11} + \psi_{16}}{\sqrt{2}}, \quad g_4^- = \frac{\psi_{10} + \psi_{13}}{\sqrt{2}}. \tag{7}$$

The components f_1 and f_2 describe the states with the total spin $s = 0$, while the rest of f's and g's — the states with $s = 1$. All f's correspond to the spin magnetic number $m_s = 0$, while all g's to $m_s = +1$ and $m_s = -1$ simultaneously, $g = \frac{1}{\sqrt{2}}[g(m_s = +1) + g(m_s = -1)]$, where the components with $m_s = \pm 1$ are given by relations

$$g_1(m_s = \pm 1) = \frac{g_1 \mp i g_4}{\sqrt{2}} = \mp i g_4(m_s = \pm 1),$$

$$g_2(m_s = \pm 1) = \frac{g_2 \pm i g_3}{\sqrt{2}} = \pm i g_3(m_s = \pm 1) \tag{8}$$

both for components "+" and "−". The components f^+ and g^+ differ from appropriate components f^- and g^- by sign of the intrinsic parity π which is described in representation (5) by the matrix

$$\eta\beta^{(1)}\beta^{(2)} = \eta\sigma_1 \times \mathbf{1} \times \sigma_1 \times \sigma_3, \tag{9}$$

where $\eta^2 = 1$. In fact, f^+ and g^+ correspond to $\pi = +\eta$ while f^- and g^- to $\pi = -\eta$. They are "large-large" or "small-small" components for $\pi = +\eta$ and "small-large" or "large--small" components for $\pi = -\eta$. Thus for states ψ with the total parity $P = \pi(-1)^l$ equal to $\pm\eta$ the components "+" and "−" are related to $(-1)^l = \pm 1$ and $(-1)^l = \mp 1$, respectively. It follows that these components can have the orbital angular momentum $l = 0, 2, 4, \ldots$ and $l = 1, 3, 5, \ldots$, respectively, or vice versa. Here $l = j$ for $s = 0$ and $l = j-1, j, j+1$ for $s = 1$ if $j > 0$ (and $l = s$ if $j = 0$ or $j = s$ if $l = 0$). The components "1" and "2" have $l = j$, while "3" and "4" have $l = j \mp 1$ mixed if $j > 0$ (and $l = 1$ if $j = 0$). All g's correspond to $s = 1$, so they vanish for $j = 0$ as g_1 and g_2 have $l = j$ and g_3 and g_4 depend linearly on g_1 and g_2.

Let us notice that in the radial equation (4) matrices $\alpha_2^{(1)}$ and $\alpha_2^{(2)}$ which mix the components $\psi_1, \ldots, \psi_8$ and $\psi_9, \ldots \psi_{16}$ appear only via their product

$$\alpha_2^{(1)}\alpha_2^{(2)} = \mathbf{1} \times \sigma_3 \times \mathbf{1} \times \mathbf{1} \tag{10}$$

which is diagonal. This is the formal reason why the system of 16 equations given in Table I splits into two separate subsystems of 8 equations containing $\psi_1, \ldots, \psi_8$ and $\psi_9, \ldots, \psi_{16}$. If $j = 0$ each of these two subsystems splits in turn into two separate subsystems of 4 equations for four f's and four g's, where all g's vanish. For f's we get in this case the following subsystems of 2 equations for $f_2^\pm$ and $f_3^\mp$ (with $j = 0$) if we eliminate $f_1^\pm$ and $f_4^\mp$ and consider the static case (where $V(\vec{r}) = V(r)$):

$$\begin{gathered} \frac{d}{dr} f_2^\pm + \tfrac{1}{2}\left[E - V - \frac{(m^{(1)} \mp m^{(2)})^2}{E-V}\right] f_3^\mp = 0, \\ -\left(\frac{d}{dr} + \frac{2}{r}\right) f_3^\mp + \tfrac{1}{2}\left[E - V - \frac{(m^{(1)} \pm m^{(2)})^2}{E-V}\right] f_2^\pm = 0. \end{gathered} \tag{11}$$

In particular, the ground bound state, if it includes the components f_1^+ and f_2^+ with $j = 0$, $l = 0$, $s = 0$ and $P = \eta$, can be calculated from the system (11) with upper signs. By iterating the first-order equations (11) we obtain for $f_2^\pm$ and $f_3^\pm$ (with $j = 0$) the separate second-order equations

$$\left\{\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + \frac{(E-V)^2 + (m^{(1)} \mp m^{(2)})^2}{(E-V)^2 - (m^{(1)} \mp m^{(2)})^2} \frac{\frac{dV}{dr}}{E-V}\frac{d}{dr} + \tfrac{1}{4}\left[(E-V)^2 - 2(m^{(1)2} + m^{(2)2}) + \left(\frac{m^{(1)2} - m^{(2)2}}{E-V}\right)^2\right]\right\} f_2^\pm = 0 \tag{12}$$

and

$$\left\{\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{2}{r^2}+\frac{(E-V)^2+(m^{(1)}\mp m^{(2)})^2}{(E-V)^2-(m^{(1)}\mp m^{(2)})^2}\frac{\frac{dV}{dr}}{E-V}\left(\frac{d}{dr}+\frac{2}{r}\right)\right.$$

$$\left.+\tfrac{1}{4}\left[(E-V)^2-2(m^{(1)2}+m^{(2)2})+\left(\frac{m^{(1)2}-m^{(2)2}}{E-V}\right)^2\right]\right\}f_3^{\pm}=0. \tag{13}$$

Either is equivalent to the system (11) in the sense of differential equations. We can confirm from Eqs (12) and (13) that the components $f_2^{\pm}$ and $f_3^{\mp}$ (with $j=0$) correspond to $l=0$ and $l=1$, respectively (and $P=\pm\eta$).

3. *Relativistic barrier*

Now, let us consider the case of equal masses, $m^{(1)}=m^{(2)}=2m$, where $m = m^{(1)}m^{(2)}/(m^{(1)}+m^{(2)})$ is the reduced mass of two particles. Then Eq. (12) with upper signs takes the form

$$\left[-\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}\right)+\frac{\frac{dV}{dr}}{V-E}\frac{d}{dr}-\frac{(E-V)^2-(4m)^2}{4}\right]f=0, \tag{14}$$

where $f=f_2^{+}$ (with $j=0$). It can be rewritten as a Schrödinger-like equation (for $j=0$, $s=0$ and $l=0$)

$$\left[-\frac{1}{2m}\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}\right)+V_{\text{eff}}-\Delta E\right]f=0, \tag{15}$$

where $\Delta E=E-4m$ and

$$V_{\text{eff}}=\frac{1}{2m}\frac{\frac{dV}{dr}}{V-E}\frac{d}{dr}-\frac{(V-E)^2}{8m}+E-2m \tag{16}$$

is an effective potential. If $|(V-\Delta E)/4m|\ll 1$, Eq. (15) goes over into the true Schrödinger equation

$$\left(\frac{\vec{p}^{\,2}}{2m}+V-\Delta E\right)\psi=0. \tag{17}$$

Eq. (14) differs from the two-body Klein-Gordon equation

$$\left[\vec{p}^{\,2}-\frac{(E-V)^2-(4m)^2}{4}\right]\psi=0 \tag{18}$$

by the additional term

$$2m V_{\text{eff}}^{\text{B}} = \frac{\frac{dV}{dr}}{V-E} \frac{d}{dr} \tag{19}$$

which turns out to be characteristic for the double Dirac equation (1). We will show that for $V > 0$ this term gives an energy-dependent relativistic barrier (between or around two Dirac particles in the state f_2^+) with a relative radius $r_0 = r_0(E)$ determined by the equation $V(r_0) = E$. We will discuss this problem in the case of potentials $V = \alpha/r$ and $V = \mu^2 r$ representing the well-known Coulombic repulsion and confining attraction, respectively. To this end it will be convenient to introduce to Eq. (14) the function $u = rf$ obtaining equation

$$\left[-\frac{d^2}{dr^2} + \frac{\frac{dV}{dr}}{V-E}\left(\frac{d}{dr} - \frac{1}{r}\right) - \frac{(E-V)^2-(4m)^2}{4}\right] u = 0. \tag{20}$$

In the case of potential $V = \pm\alpha/r$, Eq. (20) takes the form

$$\left[-\frac{d^2}{dr^2} + \left(\frac{1}{r \mp \frac{\alpha}{E}} - \frac{1}{r}\right)\left(\frac{d}{dr} - \frac{1}{r}\right) - \tfrac{1}{4}\frac{\alpha^2}{r^2} \pm \tfrac{1}{2}\frac{\alpha E}{r} - \tfrac{1}{4}E^2 + (2m)^2\right] u = 0. \tag{21}$$

This equation implies the following limiting behaviour of u

$$\text{if } r \to \infty : u \sim e^{\pm ikr}, \quad k = \tfrac{1}{2}\sqrt{E^2-(4m)^2},$$

$$\text{if } r \to 0 : u \sim r^{\gamma}, \quad \gamma = \sqrt{1-\alpha^2/4}$$

and in the case of $V = \alpha/r$

$$\text{if } r \to \frac{\alpha}{E} : u \sim \left(r - \frac{\alpha}{E}\right)^2.$$

The last property indicates that in this case a relativistic barrier appears for f_2^+ at $r = \alpha/E$. This is consistent with the fact that the effective potential (16), which is given for $V = \pm\alpha/r$ by formula

$$V_{\text{eff}} = \frac{1}{2m}\left(\frac{1}{r \mp \frac{\alpha}{E}} - \frac{1}{r}\right)\frac{d}{dr} - \frac{1}{8m}\frac{\alpha^2}{r^2} \pm \frac{1}{4m}\frac{\alpha E}{r} - \frac{1}{8m}E^2 + E - 2m, \tag{22}$$

grows to infinity for $V = \alpha/r$ if $r \to \alpha/E$ as

$$V_{\text{eff}} f \simeq \frac{1}{2m} \frac{1}{r - \frac{\alpha}{E}} \frac{d}{dr} f \simeq \frac{1}{2m} \frac{2}{\left(r - \frac{\alpha}{E}\right)^2} f. \tag{23}$$

The energy-dependent radius r_0 of this relativistic barrier is always smaller than one half of the classical radius r_c of each of two interacting particles,

$$r_0 = \frac{\alpha}{E} < \frac{\alpha}{4m} = \tfrac{1}{2} r_C, \tag{24}$$

if only $E \geqslant 4m$. For instance, in the case of two interacting electrons $r_0 \leqslant 1.41 \times 10^{-13}$ cm. Obviously, $r_0 \to 0$ if $E \to \infty$. We can see, therefore, that two Dirac particles interacting by repulsive potential $V = \alpha/r$ are in the state f_2^+ either separated ($r > r_0$) or confined ($r < r_0$) by this barrier, since V_{eff} is finite in two separate space regions $r > r_0$ and $0 < r < r_0$.[1] Notice that in the first region $V_{\text{eff}} f$ is repulsive and switches off if $r \to \infty$, while in the second region it is attractive and gives

$$V_{\text{eff}} f \simeq -\frac{1}{16m^2}\frac{\alpha^2}{r^2} f \tag{25}$$

if $r \to 0$. Thus, for extremely small r the relativistic attraction $\sim \alpha^2/r^2$ dominates. We should like to stress that the effect of relativistic barrier, appearing for repulsion $V = \alpha/r$, does not occur in Eq. (21) for attraction $V = -\alpha/r$ since then $V \neq E$.

In the case of potential $V = \mu^2 r$, Eq. (20) assumes the form

$$\left[-\frac{d^2}{dr^2} + \frac{1}{r - \frac{E}{\mu^2}}\left(\frac{d}{dr} - \frac{1}{r}\right) - \frac{\mu^4}{4}\left(r - \frac{E}{\mu^2}\right)^2 + (2m)^2\right] u = 0. \tag{26}$$

The following limiting behaviour of u is implied by Eq. (26):

$$\text{if } r \to \infty : u \sim e^{\pm i \frac{1}{4}\mu^2 r^2},$$

$$\text{if } r \to 0 : u \sim r,$$

and

$$\text{if } r \to \frac{E}{\mu^2} : u \sim \left(r - \frac{E}{\mu^2}\right)^2.$$

Thus we get for f_2^+ a relativistic barrier at $r = E/\mu^2$. It is consistent with the effective potential (16) which is given now by formula

$$V_{\text{eff}} = \frac{1}{2m}\frac{1}{r - \frac{E}{\mu^2}}\frac{d}{dr} - \frac{\mu^4}{8m}\left(r - \frac{E}{\mu^2}\right)^2 + E - 2m \tag{27}$$

---

[1] In a state corresponding to a single component of the wave function ψ, we can speak only of partial confinement, since a leakage through coupling of various components is possible, unless the confining barrier is common to all components of ψ.

and rises infinitely if $r \to E/\mu^2$ as

$$V_{\text{eff}} f \simeq \frac{1}{2m} \frac{1}{r - \frac{E}{\mu^2}} \frac{d}{dr} f \simeq \frac{1}{2m} \frac{2}{\left(r - \frac{E}{\mu^2}\right)^2} f. \tag{28}$$

If $E \geqslant 4\, m_{\text{eff}}$, where $m_{\text{eff}} > 0$ is an effective reduced mass of two particles, the energy--dependent radius r_0 of this relativistic barrier is always larger than $4m_{\text{eff}}/\mu^2$,

$$r_0 = \frac{E}{\mu^2} \geqslant \frac{4m_{\text{eff}}}{\mu^2}\,. \tag{29}$$

For example, in the case of charmed quark and antiquark interacting by potential $V = \mu^2 r$ alone, we get $r_0 \gtrsim 3\ \text{GeV}/\mu^2$, if we put $2m_{\text{eff}} = 1.5$ GeV. Evidently, $r_0 \to \infty$ if $E \to \infty$. It can be seen, therefore, that two Dirac particles interacting by attractive potential $V = \mu^2 r$ are in the state f_2^+ either confined $(r < r_0)$ or separated $(r > r_0)$ by this relativistic barrier, because V_{eff} is finite in two separate space regions $r < r_0$ and $r_0 < r < \infty$ (see footnote no 1). Notice that in the first region $V_{\text{eff}} f$ is attractive, while in the second region it is repulsive and gives

$$V_{\text{eff}} f \simeq -\frac{\mu^4}{8m} r^2 f \tag{30}$$

if $r \to \infty$. Thus, for properly large r the relativistic repulsion $\sim \mu^4 r^2$ dominates. Since the effective potential (30) does not switch off for $r \to \infty$, the mutually repulsing particles do not become free for infinite separation. In fact, they are perpetually accelerated by infinitely growing repulsion.

In the case of the more realistic confining potential given by Eq. (2) there exists also a relativistic barrier described by formula (19), because the equation $V(r_0) = E$ has a (unique) positive solution equal to (we drop the Breit-like terms)

$$r_0 = \frac{1}{2\mu^2} \left[E - V_0 + \sqrt{(E - V_0)^2 + 4\alpha_s \mu^2}\right]. \tag{31}$$

This relativistic barrier separates, as in the former case, the relative space of two Dirac particles in the state f_2^+ into an inner region of confinement $(r < r_0)$ and an outer region of infinitely increasing repulsion $(r > r_0)$ (see footnote no 1). Of course, the phenomenological confining potential (2), suitable for the nonrelativistic Schrödinger equation (17), may be physically inconsistent with the relativistic double Dirac equation (1). One should remember, however, that this presumably more exact equation goes over into the non-relativistic Schrödinger equation with the same potential V if $|(V - \Delta E)/4m| \ll 1$.

As a final remark we should like to stress that, so far, we discussed only Eq. (12) with upper signs which gave Eq. (14) in the case of equal masses $m^{(1)} = m^{(2)} = 2m$. In the same case Eq. (12) with lower signs takes the more complicated form

$$\left[-\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right) + \frac{(E-V)^2 + (4m)^2}{(E-V)^2 - (4m)^2} \frac{\frac{dV}{dr}}{V - E} \frac{d}{dr} - \frac{(E-V)^2 - (4m)^2}{4}\right] f = 0, \tag{32}$$

where now $f = f_2^-$ (with $j = 0$). Beside the singularity at the point $r = r_0$ determined by equation $V(r_0) = E$ we get here two other singularities given by equations $V(r_0) = E+4m$ and $V(r_0) = E-4m$. In the case of Coulombic potential $V = \pm\alpha/r$ we obtain beside $r_0 = \alpha/E$ two additional singularities

$$r_0 = \frac{\alpha}{E+4m}, \quad r_0 = \frac{\alpha}{E-4m} = \frac{\alpha}{\Delta E} \tag{33}$$

for repulsion $V = \alpha/r$ (and energies $E > 0$ and $E > 4m$, respectively), and one singularity

$$r_0 = \frac{\alpha}{4m-E} = \frac{\alpha}{-\Delta E} \tag{34}$$

for attraction $V = -\alpha/r$ and energies $E < 4m$ (i. e., for bound states). At these three singularities there appear relativistic barriers for f_2^-, since if $r \to r_0$ we get

$$V_{\rm eff}^{\rm B} = \frac{1}{2m}\frac{(E-V)^2+(4m)^2}{(E-V)^2-(4m)^2}\frac{\frac{dV}{dr}}{V-E}\frac{d}{dr} \simeq \frac{1}{2m}\frac{1}{r-r_0}\frac{d}{dr} \tag{35}$$

and $f \sim (r-r_0)^2$. On the other hand, at the singularity $r_0 = \alpha/E$ there is for f_2^- no relativistic barrier, because now

$$V_{\rm eff}^{\rm B} = \frac{1}{2m}\frac{(E-V)^2+(4m)^2}{(E-V)^2-(4m)^2}\frac{\frac{dV}{dr}}{V-E}\frac{d}{dr} \simeq -\frac{1}{2m}\frac{1}{r-r_0}\frac{d}{dr} \tag{36}$$

and $f \sim (r-r_0)^0$ $\left(\text{though } \frac{d}{dr} f \sim r-r_0\right)$.

It is easy to recognize from Eq. (13) that the structure of singularities for f_3^+ and f_3^- (with $j = 0$) is the same as for f_2^+ and f_2^- (with $j = 0$), respectively. However, while for f_2^+ and f_3^+ there exists only the barrier corresponding to $V(r_0) = E$, for f_2^- and f_3^- two barriers related to $V(r_0) = E+4m$ and $V(r_0) = E-4m$ develop. Thus, there is no relativistic barrier common to all components of the wave function ψ (with $j = 0$) and, consequently, no impenterable barrier for two Dirac particles, since there appears a leakage through coupling of various components.

4. *Analogy with Dirac equation*

One can notice that the singularities given by equations $V(r_0) = E+4m$ and $V(r_0) = E-4m$ (but not $V(r_0) = E$) have their analogues in the one-particle Dirac equation

$$[E_{\rm D}-(\vec{\alpha}\cdot\vec{p}+\beta m_{\rm D})-V(\vec{r})]\psi(\vec{r}) = 0 \tag{37}$$

where $V(\vec{r})$ is a phenomenological external potential which is assumed here to be central, $V(\vec{r}) = V(r)$. Then, using the Dirac representation

$$\vec{\alpha} = \vec{\sigma} \times \sigma_1, \quad \beta = \mathbf{1} \times \sigma_3 \tag{38}$$

and denoting the "large" and "small" components by

$$\psi^+ = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \psi^- = \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix} \tag{39}$$

we get after iteration of first order equations the following separate second-order radial equations

$$\left\{ \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{l(l+1)}{r^2} + \frac{\frac{dV}{dr}}{E_D \pm m_D - V}\left[\frac{d}{dr} - \frac{j_D(j_D+1) - l(l+1) - \frac{3}{4}}{r}\right] + (E_D - V)^2 - m_D^2 \right\} \psi_l^{\pm}(r) = 0, \tag{40}$$

where $\psi_l^{\pm}(r)$ are the components with $l = j_D - \frac{1}{2}$ and $l = j_D + \frac{1}{2}$ of the radial functions $\psi^{\pm}(r)$ with a given $j_D = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$, $s_D = \frac{1}{2}$ and intrinsic parity $\pi_D = \pm\eta$ (which is described by the matrix $\eta\beta$, where $\eta^2 = 1$). In Eqs (40) with labels "$\pm$" there appear the singularities related to equations $V(r_0) = E_D \pm m_D$, respectively. If $j_D = \frac{1}{2}$, Eqs (40) for $\psi_0^{\pm}$ and $\psi_1^{\pm}$ are analogues of Eqs (12) and (13) for $f_2^{\pm}$ and $f_3^{\pm}$. We can see that also in the case of Dirac equation there is no relativistic barrier common to all components of the wave function ψ. We get, therefore, no impenetrable barrier for a Dirac particle in an external potential, since there appears a leakage through coupling of various components.

5. *Conclusion*

While there is no confinement of two Dirac particles caused by relativistic barriers, the physical reality of the confinement for various wave-function components depends, of course, on the existence and character of solutions inside these barriers and also, more fundamentally, on the applicability of double Dirac equation (1) to the small-distance regions. We hope to come back to the first question elsewhere[2].

---

[2] In particular, a phenomenon analogous to Klein paradox which is a well known disease of the one--particle Dirac equation, requires special attention (cf. Ref. [7]; we learnt of this work after completion of our paper).

APPENDIX

It has been observed by Z. Otwinowski that in the case of $m^{(1)} = m^{(2)}$ the subsystem of 8 equations with upper signs in Table I splits for arbitrary j into two separate subsystems of 4 equations for $f_1^+, f_2^+, f_3^-, g_4^-$ and $g_1^+, g_2^+, g_3^-, f_4^-$, respectively. This is not the case, however, for the subsystem with lower signs. If we eliminate f_1^+, g_4^- or g_1^+, f_4^- we get

$$\frac{d}{dr} f_2^+ + \tfrac{1}{2}(E-V) f_3^- = 0,$$

$$-\left(\frac{d}{dr} + \frac{2}{r}\right) f_3^- + \tfrac{1}{2}\left[E - V - \frac{(4m)^2}{E-V} - \frac{4j(j+1)}{r^2(E-V)}\right] f_2^+ = 0, \tag{A1}$$

or

$$\left(\frac{d}{dr} + \frac{1}{r}\right) g_2^+ + \tfrac{1}{2}(E-V) g_3^- = 0,$$

$$-\left(\frac{d}{dr} + \frac{1}{r}\right) g_3^- + \tfrac{1}{2}\left[E - V - \frac{(4m)^2}{E-V} - \frac{4j(j+1)}{r^2(E-V)}\right] g_2^+ = 0, \tag{A2}$$

where $m^{(1)} = m^{(2)} = 2m$. By iterating the first-order equations (A1) or (A2) we obtain

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{j(j+1)}{r^2} + \frac{\frac{dV}{dr}}{E-V}\frac{d}{dr} + \frac{(E-V)^2-(4m)^2}{4}\right] f_2^+ = 0, \tag{A3}$$

$$\left\{\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{j(j+1)+2}{r^2}\right.$$

$$+ \frac{\left[(E-V)^2 + (4m)^2 + \frac{4j(j+1)}{r^2}\right]\frac{\frac{dV}{dr}}{E-V} - \frac{8j(j+1)}{r^3}}{(E-V)^2 - (4m)^2 - \frac{4j(j+1)}{r^2}}\left(\frac{d}{dr} + \frac{2}{r}\right)$$

$$\left. + \frac{(E-V)^2-(4m)^2}{4}\right\} f_3^- = 0, \tag{A4}$$

or

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{j(j+1)}{r^2} + \frac{\frac{dV}{dr}}{E-V}\left(\frac{d}{dr} + \frac{1}{r}\right) + \frac{(E-V)^2-(4m)^2}{4}\right] g_2^+ = 0, \tag{A5}$$

$$\left\{\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{j(j+1)}{r^2}\right.$$

$$+\frac{\left[(E-V)^2+(4m)^2+\frac{4j(j+1)}{r^2}\right]\frac{\frac{dV}{dr}}{E-V}-\frac{8j(j+1)}{r^3}}{(E-V)^2-(4m)^2-\frac{4j(j+1)}{r^2}}\left(\frac{d}{dr}+\frac{1}{r}\right)$$

$$\left.+\frac{(E-V)^2-(4m)^2}{4}\right\} g_3^- = 0. \tag{A6}$$

If $j = 0$, Eqs. (A1), (A3) and (A4) reduce to the respective Eqs. (11), (12) and (13) (in the case of $m^{(1)} = m^{(2)}$).

The components $f_1^+, f_2^+, f_3^-, g_4^-$ and $g_1^+, g_2^+, g_3^-, f_4^-$ describe in a relativistic way the states denoted by nonrelativistic spectroscopic symbols $n^1l_l^P$ and $n^3l_l^P$, respectively, where $P = \pm\eta$ corresponds to even/odd l. The remaining components $f_3^+, f_4^+, g_3^+, g_4^+, f_1^-$ ($= 0$ if $m^{(1)} = m^{(2)}$), f_2^-, g_1^- ($= 0$ if $m^{(1)} = m^{(2)}$), g_2^- describe the states $n^3l_{l\mp 1}^P$, where $P = \mp\eta$ corresponds to odd/even l. The components "+" or "−" are combinations of "large-large" and "small-small" components or "small-large" and "large-small" components respectively, In fact, the components

$$\frac{(\text{"}1+\text{"})\mp(\text{"}2+\text{"})}{\sqrt{2}} \quad \text{and} \quad \frac{(\text{"}3+\text{"})\pm(\text{"}4+\text{"})}{\sqrt{2}} \quad \text{are} \quad \begin{Bmatrix}\text{"large-large"} \\ \text{"small-small"}\end{Bmatrix},$$

while

$$\frac{(\text{"}3-\text{"})\pm(\text{"}4-\text{"})}{\sqrt{2}} \quad \text{and} \quad \frac{(\text{"}1-\text{"})\mp(\text{"}2-\text{"})}{\sqrt{2}} \quad \text{are} \quad \begin{Bmatrix}\text{"small-large"} \\ \text{"large-small"}\end{Bmatrix},$$

where the upper/lower signs correspond to the upper/lower adjectives in parentheses { }.

REFERENCES

[1] T. Appelquist, H. D. Politzer, *Phys. Rev. Lett.* **34**, 43 (1975); A. De Rújula, S. L. Glashow, *Phys. Rev. Lett.* **34**, 46 (1975); T. Appelquist et al., *Phys. Rev. Lett.* **34**, 365 (1975); E. Eichten et al., *Phys. Rev. Lett.* **34**, 369 (1975).

[2] For a recent review cf.: J. D. Jackson, in *Proceedings of European Conference on Particle Physics*, Budapest 1977 (to appear); K. Gottfried, in *Proceedings of 1977 International Symposium on Lepton and Photon Interactions at High Energies*, ed. by F. Gutbrod, Hamburg 1977.

[3] W. Królikowski, J. Rzewuski, *Nuovo Cimento* **2**, 203 (1955).

[4] W. Królikowski, J. Rzewuski, *Nuovo Cimento* **3**, 260 (1956); **4**, 1212 (1956); **25B**, 739 (1975).

[5] W. Królikowski, J. Rzewuski, *Nuovo Cimento* **4**, 975 (1956); *Acta Phys. Pol.* **15**, 321 (1956).

[6] W. Królikowski, J. Rzewuski, *Acta Phys. Pol.* **B7**, 487 (1976); W. Kluźniak, W. Królikowski, J. Rzewuski, *Acta Phys. Pol.* **B9**, 43 (1978); cf. also Erratum and Addendum to this paper, *Acta Phys. Pol.* **B9**, No 7 (1978).

[7] H. Suura, *Phys. Rev. Lett.* **38**, 636 (1977).