STATIC SOLUTIONS OF EINSTEIN'S FIELD EQUATIONS FOR CHARGED SPHERES OF FLUID

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Exact interior solutions to Einstein's field equations for charged static spheres of fluid are obtained in terms of two parameters. For various values of the parameters we generate both previously known solutions and a number of new ones. It is hoped that the investigation may be of some help in connection with studies of stellar structure.

1. Introduction

Several methods of approach have been used by many authors to discuss spherically symmetric relativistic fluid spheres consisting of perfect fluid. For example, the problem of constructing a static model sphere of perfect fluid is usually solved by numerical methods using the Tolman-Oppenheimer-Volkoff (Tolman, 1939; Oppenheimer and Volkoff, 1939; Wheeler et al., 1973) equations with an equation of state specified. The complicated and nonlinear character of the Einstein equations has severely limited the number of analytic solutions to only a small number.

In this paper we generate a two-parameter analytic solutions to Einstein's field equations for charged static spheres of fluid. We demonstrate that for various values of the parameters and specific choices of the constants of integration several previously known results are obtained. It is hoped that some of these solutions may be of use in the study of the internal constitution of stars.

2. Solutions of the field equations

The appropriate line element for a static spherically symmetric system is

$$ds^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - r^2(d\Theta^2 + \sin^2 \Theta d\Phi^2), \tag{2.1}$$

where ν and λ are functions of r only which vanish as $r \to \infty$. The field equations may now

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be written in the fσrm (Nduka, 1976)

$$\tau' + f(r)\tau = g_0(r), \tag{2.2}$$

$$8\pi\varrho - \frac{Q^2}{r^4} = \frac{1}{r^2} - e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right), \tag{2.3}$$

$$8\pi p + \frac{Q^2}{r^4} = e^{-\lambda}\left(\frac{1}{r^2} + \frac{\nu'}{r}\right) - \frac{1}{r^2}, \tag{2.4}$$

where $\tau(r) = e^{-\lambda}$, $f(r) = -2(\gamma + r\gamma' - r^2\gamma'')/r(r\gamma' + \gamma)$, $g_0(r) = -(2r^2\gamma + 4\gamma Q^2)/r^3(r\gamma' + \gamma)$, $\gamma = e^{\nu/2}$ and Q is the charge up to radius r.

Equation (2.2) has the immediate solution

$$\tau = e^{-F(r)}\{\int^r e^{F(r')}g(r')dr' + \int^r e^{F(r')}g_1(r')dr' + C\} \tag{2.5}$$

with $g(r) = -2\gamma/r(r\gamma' + \gamma)$, $g_1(r) = -4\gamma Q^2/r^3(r\gamma' + \gamma)$, C is a constant and $F(r) = \int^r f(r')dr'$.

For given $\gamma(r)$ equation (2.5) together with equations (2.3) and (2.4) represent all solutions for charged static spherically symmetric spheres of fluid. Not all such solutions will be physically reasonable. Only a subclass of these solutions, corresponding to certain functions $\gamma(r)$, will admit a physically meaningful solution. Thus the choice of $\gamma(r)$ is critical for this problem.

3. Specific analytic solution

Our aim is to find a solution of equation (2.5) that is simple. Equation (2.5) can easily be evaluated if we set $Q = (K/2)^{1/2}r$ and $\gamma(r)$ is chosen to satisfy Euler's equation

$$r^2\gamma'' + \sigma r\gamma' + \beta\gamma = 0 \tag{3.1}$$

with K a parameter. If in equation (3.1) we put $\gamma = r^S$, we find two values of S:

$$S_1 = -\frac{(\sigma-1)}{2} + \tfrac{1}{2}\sqrt{(\sigma-1)^2 - 4\beta}, \qquad S_2 = -\frac{(\sigma-1)}{2} - \tfrac{1}{2}\sqrt{(\sigma-1)^2 - 4\beta}. \tag{3.2}$$

In our own case $\sigma = -1$, and to obtain a solution as simple as possible, we find that we must choose

$$\beta = 1 - \alpha^2, \qquad 0 < \alpha \leqslant 1. \tag{3.3}$$

For this particular choice of β, the solution of equation (3.1) takes the simple form

$$\gamma(r) = Ar^{S_1} + Br^{S_2}, \qquad r > 0, \tag{3.4}$$

where now $S_1 = 1 + \alpha$ and $S_2 = 1 - \alpha$.

Equations (2.5) and (3.4) may now be used to find $\tau(r)$ in the form

$$\tau(r) = \frac{1+K}{(2-\alpha^2)} + Cr^{-2\alpha\eta}\{(2+\alpha)Ar^{2\alpha} + (2-\alpha)B\}^{\eta-\xi} \tag{3.5}$$

where $\xi = (\alpha^2 - 2)/\alpha(2+\alpha)$, $\eta = (\alpha^2 - 2)/\alpha(2-\alpha)$.

It now follows from equations (2.3), (2.4), (3.4) and $\tau(r) = e^{-\lambda}$ that

$$8\pi p r^2 = \tau(r)\left[(2\alpha+3)Ar^{2\alpha}+(3-2\alpha)B\right](Ar^{2\alpha}+B)^{-1} - \frac{(2+K)}{2} \tag{3.6}$$

and

$$8\pi \varrho r^2 = (2+K)/2-\tau(r)-2[(2-\alpha^2)\tau-(1+K)]\,(Ar^{2\alpha}+B) \times[(2+\alpha)Ar^{2\alpha}+(2-\alpha)B]^{-1}. \tag{3.7}$$

The constants A, B, and C are specified by matching the solution to the exterior Nordstrom–-Reissner solution for a mass M_0, radius r_0 and charge $Q_0 = (K/2)^{1/2}r_0$. We obtain

$$A = \frac{(1-2\varepsilon+K/2)^{-1/2}[(3-2\alpha)\varepsilon-(1-\alpha)(1+K/2)]r_0^{-S_1}}{2\alpha}, \tag{3.8}$$

$$B = \frac{(1-2\varepsilon+K/2)^{-1/2}[(1+\alpha)(1+K/2)-(3+2\alpha)\varepsilon]r_0^{-S_2}}{2\alpha}, \tag{3.9}$$

$$C = \frac{[1-2\varepsilon(2-\alpha)-\alpha^2(1+K/2)]\,[(1-2\varepsilon+K/2)^{-1/2}(1-\varepsilon+K/2)]^{\xi-\eta}r_0^{-\delta}}{(2-\alpha^2)}, \tag{3.10}$$

where

$$\varepsilon = M_0/r_0 \quad \text{and} \quad \delta = 6(2-\alpha^2)/(4-\alpha^2).$$

4. *Properties of the solution*

We note that both the density $\varrho(r)$ and the pressure $p(r)$ are singular at the origin. However, the ratio $p(0)/\varrho(0)$ given by

$$\frac{p(0)}{\varrho(0)} = \frac{2(1+K)(3-2\alpha)-(2+K)(2-\alpha^2)}{(2+K)(2-\alpha^2)-2(1+K)} \tag{4.1}$$

remains finite there, depending only on the parameters α and K. Hence, whereas $p(0)/\varrho(0)$ is independent of the mass and radius of the fluid, it does depend on the charge.

For particular values of the parameters α and K several previously known results are obtained. Thus the case for which $\alpha = 1$, $K = 0$ was treated by Adler (Adler, 1974). Tolman's solutions number I, number V and number VI are obtained by putting $K = 0$ and adjusting both the parameter α and the constants of integration A and B. If we put $K = 0$, our solutions give the static solutions of Einstein's field equations for uncharged spheres of fluid.

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