ON THE NON-SYMMETRIC UNIFIED FIELD THEORY

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It is shown that the common presentation of Einstein's non-symmetric unified field theory involves a misconception. The theory implies a different approach to physics than General Relativity. The revised interpretation of its structure raises the problem of determination of the metric tensor. A solution of this problem is proposed and it is shown that for a spherically symmetric field it restricts significantly the class of possible solutions of the field equations.

1. *Introduction*

The purpose of this work is to review the foundations of Einstein's non-symmetric unified field theories (Refs [1–3] which have undergone considerable change since the theories were first proposed. It has been shown previously (Ref. [4]) that there are only two theories of this type represented respectively by the strong and weak field equations. The status of the former, that is, its physical meaning is still uncertain and we shall concern ourselves only with the latter. In any case, the difference between the two sets of field equations for a spherically symmetric field reduces (Ref. [5]) to the vanishing or otherwise, of a constant of integration. Already while Einstein was developing his theory, logical objections were raised which seemed to invalidate the possibility of its acceptance as a unified model of the gravitational and electromagnetic fields. Several authors appeared to show (Refs [6–8]) that the equations of motion of a test particle as derived from the field equations did not contain, as they should if the particle were charged, any terms corresponding to a Lorentz force. It seemed also (Ref. [9]) that a spherically symmetric point charge, that is Coulomb law, could not exist within the context of the theory. These objections have now been eliminated since they arose out of an incorrect identification of the electromagnetic field tensor. As early as 1957, it was shown by Treder (Ref. [10]) that the Lorentz force would come out of the field equations if the electromagnetic potential were a solution of the biharmonic equation. Treder, however, did not give any physical explanation why one should be entitled to modify the potential in this way. Complete resolution of this problem as well as of the Tiwari-Pant paradox was obtained later

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(Refs [11, 12]). The outcome of the work of Russell and of the present author is that, notwithstanding the still elusive empirical confirmation, the nonsymmetric unified field theory is again a possible extension of General Relativity encompassing gravitation and electromagnetism. There are also strong philosophical reasons (Ref. [13]) to suspect that it is valid.

It is therefore important to be quite clear as to the logical structure of the theory, as to the assumptions and principles laid down, and as to the relationship between the model and the physical world it purports to describe. These turn out to be quite different to the fundamental principles of General Relativity. The difference, in fact, leads to a problem solution of which constitutes the main mathematical part of this work. This is construction of the metric within the non-symmetric theory.As we shall see, the solution of the problem leads to a restriction on the class of possible solutions of the field equations themselves.

2. *Logical structure of the theory*

The field equations we are considering are the weak field equations of Einstein and Straus (Ref. [2]), or in their notation adopted throughout

$$g_{\underset{+}{\mu}\underset{-}{\nu},\lambda} = 0, \qquad R_{\underline{\mu\nu}} = 0, \qquad R_{\underset{\vee}{\mu\nu},\lambda} = 0, \qquad \Gamma_\mu = 0, \tag{1}$$

where

$$R_{\mu\nu} = -\Gamma^{\sigma}_{\mu\nu,\sigma} + \Gamma^{\sigma}_{\mu\sigma,\nu} + \Gamma^{\sigma}_{\mu\varrho}\Gamma^{\varrho}_{\sigma\nu} - \Gamma^{\sigma}_{\mu\nu}\Gamma^{\varrho}_{\sigma\varrho},$$

is the generalized Ricci tensor, $\Gamma^{\lambda}_{\mu\nu}$, the non-symmetric affine connection and $g_{\mu\nu}$, "the non--symmetric generalized metric "tensor. The latter is a misnomer and for reasons set out below I prefer to call it the fundamental tensor of the unified field theory. In order to appreciate the meaning of the non-symmetric generalization we must recall the basic assumption of General Relativity. It is postulated there that a Riemannian V_4 shall be the model of the world, its curvature representing gravitational properties of matter. In a V_4, $g_{\mu\nu}$ is the metric tensor and the affine connection is of course, given by the usual Christoffel brackets. The ten components of $g_{\mu\nu}$, that is the metric properties of space and, by hypothesis, of gravity are determined by the field equation. Selection of gravitation as the only "geometrised" field is the essence of the Principle of Equivalence.

The latter is discarded in the unified field theory in favour of the Principle of Hermitian or Transposition Invariance which, as Einstein pointed out, is the only means of restricting to manageable proportions, the freedom of choice of possible field equations. The Principle has an immediate physical interpretation as an expression of charge conjugation invariance but unlike the Principle of Equivalence it does not inform us about the metric properties of the space-time. As is well known the last of the equations (1) ensure that the weak field equations are transposition invariant.

The consequence of this replacement of equivalence, and this is not widely realised, is that the unified field theory splits into two parts which we may call respectively physics and geometry. The only a priori geometrical quantities are the sixty four components $\Gamma^{\lambda}_{\mu\nu}$ of the affine connection in terms of which the last three of the equations (1) are written down. The affine connection tells us how to compare vectors at different points of the space-time and determines both the Riemann-Christoffel curvature tensor and, of course, its contractions and therefore the form of these equations.

The first part, "physics", is contained in the sixteen components $g_{\mu\nu}$ of the fundamental tensor. The fact that in a fourdimensional manifold there are only sixteen of them restricts a priori the number of physical fields which can be unified. Indeed, discovery of a macroscopic, that is not subject to quantum mechanical concepts, field other than gravitation and electromagnetism, would upset Einstein's theory and it would almost certainly imply an increase in the dimensionality of the manifold. Now, as in General Relativity, Mme Tonnelat has shown (Ref. [14]) that under not very restrictive conditions ($g(g-2) \neq 0$, $g \equiv \det(g_{\mu\nu})$), the first of equations (1) determines $\Gamma^{\lambda}_{\mu\nu}$ uniquely in terms of $g_{\mu\nu}$ and its first derivatives. This, however, does not assign to the physical $g_{\mu\nu}$ a geometrical meaning because there is no geometrical meaning yet to be assigned to it.

A quasi-geometrisation of physics, or rather of macrophysics of gravitation and electromagnetism, which after all is the goal of the theory, is achieved only after substitution of Tonnelat solution for $\Gamma^{\lambda}_{\mu\nu}$ into the remaining field equation and their subsequent solution under some preassigned symmetry conditions. Even then it produces only the affine structure of the now physical world. Whatever the interpretation of $g_{\underset{\vee}{\mu\nu}}$ (and it certainly cannot be the electromagnetic field tensor, Refs [11, 12]), to postulate that the symmetric part $g_{\underline{\mu\nu}}$ of the fundamental tensor is the metric of the space-time is equivalent to making an ad hoc hypothesis foreign to the spirit of the theory. I claim that even though frequently made, this hypothesis or any of its variants (e. g., Refs [15, 16]) are unwarranted.

As we see, the above considerations imply that Einstein's non-symmetric theory is affine in the sense that it only permits an affine geometrical structure to the hypothetical model of the world. On the other hand, it is very different from Schrödinger's "Purely Affine Theory" (Ref. [16]) in spite of identical notation responsible for considerable confusion between them. If the latter is to be considered then, strictly speaking, all sixty four $\Gamma^{\lambda}_{\mu\nu}$'s should be assigned physical meaning. This at present is hard to visualise. Einstein's theory saves the situation by having only sixteen functions to be identified with physically meaningful quantities.

However, it leaves unanswered the question of what are the metrical properties of the world. It is difficult to think of macrophysics without reference to how distances between points are to be measured especially in view of the success of basically metrised General Relativity. In the sequel therefore, we shall attempt to formulate the conditions which should lead to an unequivocal determination of the metric, if one wishes to say so, of the background Riemannian space of unified-field structure. We shall also carry out such determination in the spherically symmetric, static case.

3. *Definition of the metric*

The hypothesis that $g_{\underline{\mu\nu}}$ is the metric as well as the proposals of Wyman (Ref. [15], with the metric explicitly dependent on $g_{\underset{\vee}{\mu\nu}}$) and of Schrödinger (Ref. [16]) all satisfy the following conditions. If the metric tensor is denoted by $a_{\mu\nu}$ which is non-singular ($a \equiv \det$

$a_{\mu\nu} \neq 0$) then

$$a_{\mu\nu} = a_{\mu\nu}(\underline{g}_{\mu\nu}, \underset{\vee}{g}_{\mu\nu}) = a_{\nu\mu},$$

and

$$a_{\mu\nu}(\underline{g}_{\mu\nu}, 0) = \underline{g}_{\mu\nu}. \tag{2}$$

We can also require $a_{\mu\nu}$ to depend also on $\Gamma^{\lambda}_{\mu\nu}$ in such a way that when $\underset{\vee}{g}_{\mu\nu} = 0 = \underset{\vee}{\Gamma}^{\lambda}_{\mu\nu}$, it reduces to the metric of general relativistic V_4. This preserves complete generality and in either form we can adopt conditions (2) as necessary for any second rank tensor which is to be regarded as the metric tensor of the physical space-time.

We now seek the most general expression which would enable us to obtain $a_{\mu\nu}$, possibly as a result of calculation, and which would ensure that conditions (2) are satisfied. Now, according to the considerations of the preceding section the only geometrical quantities in the theory are the affine components $\Gamma^{\lambda}_{\mu\nu}$, of which the symmetric $\underline{\Gamma}^{\lambda}_{\mu\nu}$ constitute a connection while, of course, $\underset{\vee}{\Gamma}^{\lambda}_{\mu\nu}$ is a tensor. One would expect that the essentially geometrical $a_{\mu\nu}$'s should be determined in terms of them. Apart from deliberate complexifications (which would probably be unsolvable in any case) the simplest (and as Einstein clearly understood, simplicity must be retained as a quide in constructing a theory which does not arise from empirical necessity) assumption we can make is that $a_{\mu\nu}$ should be given by

$$\tfrac{1}{2}\, a^{\lambda\sigma}(a_{\sigma\nu,\mu}+a_{\mu\sigma,\nu}-a_{\mu\nu,\sigma}) \equiv \left\{ {\lambda \atop \mu\nu} \right\}_a = \underline{\Gamma}^{\lambda}_{\mu\nu}. \tag{3}$$

or, equivalently, by

$$a_{\mu\nu,\lambda}-\underline{\Gamma}^{\sigma}_{\mu\lambda}a_{\sigma\nu}-\underline{\Gamma}^{\sigma}_{\nu\lambda}a_{\mu\sigma} = 0. \tag{4}$$

Equations (4) must be regarded as differential equations for $a_{\mu\nu}$. There are forty equations (4) for the ten functions $a_{\mu\nu}$. However, the integrability conditions

$$a_{\mu\nu,\lambda\kappa} = a_{\mu\nu,\kappa\lambda} \tag{5}$$

of equations (4) are easily seen to be equivalent to

$$(\underline{\Gamma}^{\sigma}_{\mu\lambda,\kappa}-\underline{\Gamma}^{\sigma}_{\mu\kappa,\lambda}-\underline{\Gamma}^{\varrho}_{\mu\kappa}\underline{\Gamma}^{\sigma}_{\varrho\lambda}+\underline{\Gamma}^{\varrho}_{\mu\lambda}\underline{\Gamma}^{\sigma}_{\varrho\kappa})a_{\sigma\nu} = (\underline{\Gamma}^{\sigma}_{\nu\kappa,\lambda}-\underline{\Gamma}^{\sigma}_{\nu\lambda,\kappa}-\underline{\Gamma}^{\varrho}_{\nu\lambda}\underline{\Gamma}^{\sigma}_{\varrho\kappa}+\underline{\Gamma}^{\varrho}_{\nu\kappa}\underline{\Gamma}^{\sigma}_{\varrho\lambda})a_{\mu\sigma}.$$

or

$$R^{\sigma}_{\mu\lambda\kappa}a_{\sigma\nu} = R^{\sigma}_{\nu\kappa\lambda}a_{\mu\sigma}, \tag{6}$$

where $R^{\sigma}_{\mu\lambda\kappa}$ is the Riemann-Christoffel tensor constructed with the symmetric part $\underline{\Gamma}^{\lambda}_{\mu\nu}$ of the affine connection. But in view of the assumption (3), it is also the Riemann-Christoffel tensor of the background Riemannian space in which $a_{\mu\nu}$ is the metric tensor used for raising and lowering of tensor indices. Hence, equation (6) is equivalent to the well known symmetry relation of the Riemann-Christoffel tensor

$$R_{\nu\mu\lambda\kappa} = R_{\mu\nu\kappa\lambda}. \tag{7}$$

Since this is satisfied, it follows that the differential equations (4) are always formally integrable.

Moreover, equations

$$g_{\underset{+}{\mu}\underset{-}{\nu};\lambda} = g_{\mu\nu,\lambda} - \Gamma^{\sigma}_{\mu\lambda} g_{\sigma\nu} - \Gamma^{\sigma}_{\lambda\nu} g_{\mu\sigma} = 0$$

reduce, of course, to the corresponding equations defining the Christoffel brackets when skew symmetric parts of $g_{\mu\nu}$ and of $\Gamma^{\lambda}_{\mu\nu}$ vanish. We then get

$$\left\{ {\lambda \atop \mu\nu} \right\}_a = \left\{ {\lambda \atop \mu\nu} \right\}_{g_{\mu\nu}} \tag{8}$$

which, apart from an irrelevant constant factor, imply that $a_{\mu\nu}(g_{\mu\nu}, 0) = g_{\underline{\mu\nu}}$, as required by our defining conditions (2). We now proceed to determine the metric tensor for a spherically symmetric field.

4. *Reduction of equations determining the metric*

It was shown by Papapetrou (Ref. [17]) that the general form of a spherically symmetric, time independent fundamental tensor is

$$g_{11} = -\alpha, \quad g_{22} = -\beta = g_{33}\operatorname{cosec}^2\theta, \quad g_{44} = \sigma,$$
$$g_{23} = -g_{32} = f\sin\theta, \quad g_{14} = -g_{41} = w, \tag{9}$$

where α, β, σ, f and w are functions of $x^1 = r$ only, and the first three are strictly positive. Then also

$$0 < -g = \alpha\sigma\left(1 - \frac{w^2}{\alpha\sigma}\right)(f^2+\beta^2)\sin^2\theta. \tag{10}$$

Following Vanstone (Ref. [5]) we write

$$U = 1 - \frac{w^2}{\alpha\sigma} > 0, \quad \varrho^2 = f^2+\beta^2 = e^{2A},$$
$$y = \sigma U, \quad x = \frac{\varrho^2}{\alpha}, \quad A' = \frac{\varrho'}{\varrho}, \quad \tan B = \frac{\beta}{f}. \tag{11}$$

Dashes will throughout denote differentiation with respect to r. Mme Tonnelat showed (Ref. [14]) that in this case, the non-vanishing components of the affine connection are

$$\Gamma^1_{11} = \frac{\alpha'}{2\alpha}, \quad \Gamma^1_{22} = \Gamma^1_{33}\operatorname{cosec}^2\theta = \frac{1}{2\alpha}(fB'-\beta A'), \quad \Gamma^1_{44} = \frac{\sigma}{2\alpha}(\ln yU)',$$
$$\Gamma^2_{33} = -\sin\theta\cos\theta, \quad \Gamma^3_{\underline{23}} = \cot\theta, \quad \Gamma^2_{\underline{12}} = \Gamma^3_{\underline{13}} = \tfrac{1}{2}A' = \tfrac{1}{2}\frac{\varrho'}{\varrho}, \tag{12}$$
$$\Gamma^2_{\underline{34}} = -\Gamma^3_{\underline{24}}\sin^2\theta = \frac{wB'}{2\alpha}\sin\theta, \quad \Gamma^4_{\underline{14}} = \tfrac{1}{2}\frac{y'}{y},$$

$$\Gamma^{1}_{\underset{\vee}{23}} = \frac{1}{2\alpha}(\beta B' + fA')\sin\theta, \qquad \Gamma^{2}_{\underset{\vee}{31}} = \Gamma^{3}_{\underset{\vee}{12}}\sin^2\theta = -\frac{B'}{2}\sin\theta,$$

$$\Gamma^{1}_{\underset{\vee}{14}} = -2\Gamma^{2}_{\underset{\vee}{24}} = -2\Gamma^{3}_{\underset{\vee}{34}} = \frac{w\varrho'}{\alpha\varrho}\left(= \frac{\sigma}{2w}(\ln U)'\right).$$

Vanishing of Γ_μ, or equivalently of $(\sqrt{-g}\, g^{\underset{\vee}{\mu\nu}})_{,\nu}$, immediately gives (Ref. [5])

$$\frac{w^2}{\alpha\sigma} = 1 - U = \frac{k^2}{k^2+\varrho^2}, \qquad U = \frac{\varrho^2}{k^2+\varrho^2}, \tag{13}$$

where k is a real constant of integration; equation (13) is one integral of the field equations (1). We need (11) and (12) in the sequel.

For the sake of completeness we shall now write out equations (4) in full in the case corresponding to Tonnelat's solution (12). The coordinate system is $x^\mu = (x^1, x^2, x^3, x^4) = (r, \theta, \phi, t)$. Thus

$$a_{11,1} = 2\Gamma^{1}_{11}a_{11}; \qquad a_{11,2} = 2\Gamma^{2}_{\underline{12}}a_{12};$$

$$a_{11,3} = 2\Gamma^{3}_{\underline{13}}a_{13}; \qquad a_{11,4} = 2\Gamma^{4}_{\underline{14}}a_{14};$$

$$a_{12,1} = (\Gamma^{1}_{11} + \Gamma^{2}_{\underline{12}})a_{12}; \qquad a_{12,2} = \Gamma^{2}_{\underline{12}}a_{22} + \Gamma^{1}_{22}a_{11};$$

$$a_{12,3} = \Gamma^{3}_{\underline{13}}a_{23} + \Gamma^{3}_{\underline{23}}a_{13}; \qquad a_{12,4} = \Gamma^{4}_{\underline{14}}a_{24} + \Gamma^{3}_{\underline{24}}a_{13};$$

$$a_{13,1} = (\Gamma^{1}_{11} + \Gamma^{3}_{\underline{13}})a_{13}; \qquad a_{13,2} = \Gamma^{2}_{12}a_{23} + \Gamma^{3}_{\underline{23}}a_{13};$$

$$a_{13,3} = \Gamma^{3}_{\underline{13}}a_{33} + \Gamma^{1}_{33}a_{11} + \Gamma^{2}_{33}a_{12}; \qquad a_{13,4} = \Gamma^{4}_{\underline{14}}a_{34} + \Gamma^{2}_{\underline{34}}a_{12};$$

$$a_{14,1} = (\Gamma^{1}_{11} + \Gamma^{4}_{\underline{14}})a_{14}; \qquad a_{14,2} = \Gamma^{2}_{\underline{12}}a_{24} + \Gamma^{3}_{\underline{24}}a_{13};$$

$$a_{14,3} = \Gamma^{3}_{\underline{13}}a_{34} + \Gamma^{2}_{\underline{34}}a_{12}; \qquad a_{14,4} = \Gamma^{4}_{\underline{14}}a_{44} + \Gamma^{1}_{44}a_{11};$$

$$a_{22,1} = 2\Gamma^{2}_{\underline{12}}a_{22}; \qquad a_{22,2} = 2\Gamma^{1}_{22}a_{12};$$

$$a_{22,3} = 2\Gamma^{3}_{\underline{23}}a_{23}; \qquad a_{22,4} = 2\Gamma^{3}_{\underline{24}}a_{23};$$

$$a_{23,1} = 2\Gamma^{2}_{\underline{12}}a_{23}; \qquad a_{23,2} = \Gamma^{1}_{22}a_{13} + \Gamma^{3}_{\underline{23}}a_{23}; \tag{4'}$$

$$a_{23,3} = \Gamma^{3}_{\underline{23}}a_{33} + \Gamma^{1}_{33}a_{12} + \Gamma^{2}_{33}a_{22}; \qquad a_{23,4} = \Gamma^{3}_{\underline{24}}a_{33} + \Gamma^{2}_{\underline{34}}a_{22};$$

$$a_{24,1} = (\Gamma^{2}_{\underline{12}} + \Gamma^{4}_{\underline{14}})a_{24}; \qquad a_{24,2} = \Gamma^{1}_{22}a_{14} + \Gamma^{3}_{\underline{24}}a_{23};$$

$$a_{24,3} = \Gamma^{3}_{\underline{23}}a_{34} + \Gamma^{2}_{\underline{34}}a_{22}; \qquad a_{24,4} = \Gamma^{3}_{\underline{24}}a_{34} + \Gamma^{1}_{44}a_{12};$$

$$a_{33,1} = 2\Gamma^{3}_{\underline{13}}a_{33}; \qquad a_{33,2} = 2\Gamma^{3}_{\underline{23}}a_{33};$$

$$a_{33,3} = 2\Gamma^1_{33}a_{13}+2\Gamma^2_{33}a_{23}; \qquad a_{33,4} = 2\underline{\Gamma}^2_{34}a_{23};$$

$$a_{34,1} = (\underline{\Gamma}^3_{13}+\underline{\Gamma}^4_{14})a_{34}; \qquad a_{34,2} = \underline{\Gamma}^3_{23}a_{34}+\underline{\Gamma}^3_{24}a_{33};$$

$$a_{34,3} = \Gamma^1_{33}a_{14}+\Gamma^2_{33}a_{24}+\underline{\Gamma}^2_{34}a_{23}; \qquad a_{34,4} = \underline{\Gamma}^2_{34}a_{24}+\Gamma^1_{44}a_{13};$$

$$a_{44,1} = 2\underline{\Gamma}^4_{14}a_{44}; \qquad a_{44,2} = 2\underline{\Gamma}^3_{24}a_{34};$$

$$a_{44,3} = 2\underline{\Gamma}^2_{34}a_{24}; \qquad a_{44,4} = 2\Gamma^1_{44}a_{14}.$$

Resolution of these equations in general is tedious and not very profitable especially since they will have to be investigated in full in the generalised non-symmetric theory based on the field equations

$$R_{\underline{\mu\nu}} = 0, \qquad R_{\underline{\mu\nu}} = \tfrac{1}{3}(\Gamma_{\mu,\nu}-\Gamma_{\nu,\mu}),$$

$$g_{\underset{+}{\mu}\underset{-}{\nu};\lambda}-\tfrac{2}{3}(g_{\mu\nu}\Gamma_\lambda+g_{\mu\lambda}\Gamma_\nu) = 0, \tag{14}$$

which is beyond the scope of the present work. Accordingly, we shall confine ourselves to the case when $a_{\mu\nu}$ is diagonal and its components depend on r and θ only.

The equations (4′) then immediately give

$$a_{11} = a_0\alpha, \qquad a_{22} = a_{33}\operatorname{cosec}^2\theta = b_0\varrho, \qquad a_{44} = \frac{y}{y_0}, \tag{15}$$

where a_0, b_0 and y_0 are non-zero constants together with the algebraic conditions

$$\underline{\Gamma}^2_{12}a_{22}+\Gamma^1_{22}a_{11} = 0, \qquad \underline{\Gamma}^3_{13}a_{33}+\Gamma^1_{33}a_{11} = 0,$$

$$\underline{\Gamma}^4_{14}a_{44}+\Gamma^1_{44}a_{11} = 0$$

and

$$\underline{\Gamma}^2_{34} = 0, \tag{16}$$

the remaining equations reducing to identities.

The second of equations (16) is automatically satisfied while the others become respectively

$$b_0\varrho'+a_0\left(fB'-\beta\frac{\varrho'}{\varrho}\right) = 0, \qquad y'+a_0y_0\sigma(\ln yU)' = 0, \qquad wB' = 0. \tag{17}$$

The solution of these equations represents a restriction on the general solution of the field equations (obtained by Vanstone) required by our hypothesis that $a_{\mu\nu}$ determined by equations (4) should be the metric of the background Riemannian space. We have asserted that it is through the latter that measurement of distance between points of the space-time manifold becomes possible in the non-symmetric theory. The second of equations (17) can be integrated at once since $\sigma = y/U$, and gives

$$y = \sigma U = y_1\left(1+\frac{a_0 y_0}{U}\right), \tag{18}$$

where y_1 is yet another (positive) constant of integration. The profligacy of integration constants can clearly be reduced but it is advantageous to retain them at this stage. Also, the last of equations (17) requires that either $w = 0$ or $f \propto \beta$ ($B' = 0$, or both $w, B' = 0$). We must now check whether (18), and the result of integrating the first of equations (17) under these conditions are compatible with Vanstone's general solution of the weak field equations and what further restrictions they imply.

5. *Vanstone's solution*

Vanstone's solution of equations (1), although general, is expressed (Ref. [5]) in a form inconvenient for our purpose. Let us therefore consider it again. The field equations which are not identically satisfied are

$$R_{11} = 0, \qquad R_{22} = 0, \qquad R_{44} = 0, \qquad R_{\underset{\vee}{23}} = c\sin\theta, \tag{19}$$

where c is a real constant. θ dependence is not significant, so let us write $\Gamma^1_{\underset{\vee}{23}}$ for $\Gamma^1_{\underset{\vee}{23}}$ cosec θ. Then the R_{22} and R_{23} equations become respectively

$$(\Gamma^1_{22})'+\tfrac{1}{2}(\ln \alpha y)'\Gamma^1_{22}+B'\Gamma^1_{\underset{\vee}{23}}+1 = 0,$$

$$(\Gamma^1_{\underset{\vee}{23}})'+\tfrac{1}{2}(\ln \alpha y)'\Gamma^1_{\underset{\vee}{23}}-B'\Gamma^1_{22}+c = 0, \tag{20}$$

These may be written as equations in A and B:

$$B''+\left(\frac{x'}{x}+\frac{y''}{y'}\right)B'+\frac{2}{x}(f+c\beta) = 0,$$

and

$$A''+\left(\frac{x'}{x}+\frac{y''}{y'}\right)A'+\frac{2}{x}(cf-\beta) = 0. \tag{21}$$

Similarly, the R_{11} and R_{44} equations give

$$2A''-A'^2+B'^2-2\frac{y''}{y'}A' = 0,$$

and

$$w^2 = k^2\frac{y}{x} = \frac{k^2}{\lambda^2}y'^2, \tag{22}$$

where we have put λ^2 instead of Vanstone's λ. We shall consider only the case $\lambda^2 \neq 0$ unless otherwise dictated by the requirements of Section 4. It is now readily seen that if we write $c = \tan\varepsilon$, $y = e^s$, $D = s+A$ and $C = B-\varepsilon$, and regard s instead of r as the independent variable, the field equations to be solved become

$$\frac{d^2C}{ds^2} + \mu e^D \cos C = 0 = \frac{d^2D}{ds^2} - \mu e^D \sin C, \tag{23}$$

where $\mu = \frac{2}{\lambda^2} \sec\varepsilon$, and

$$2\frac{d^2D}{ds^2} = \left(\frac{dD}{ds}\right)^2 - \left(\frac{dC}{ds}\right)^2 - 1. \tag{24}$$

Letting now $Z = C+iD$, equations (23) become as in Vanstone's discussion, a single complex equation

$$\frac{d^2Z}{ds^2} + \mu e^{-iZ} = 0, \tag{25}$$

in which however, s and μ are real. Writing further $v = 2i\mu$, the general solution becomes

$$\exp(iZ) = \frac{v}{2\zeta_0}(1-\cos\sqrt{\zeta_0}(s-s_0)), \tag{26}$$

where ζ_0 ans s_0 are complex constants of integration. Let now $\sqrt{\zeta_0} = p+iq$, $s_0\sqrt{\zeta_0} = s_1+is_2$, where p, q, s_1, s_2 are real, and

$$\lambda_1 = \frac{2pq\mu}{(p^2+q^2)^2} = \mu_1 \sin\delta, \quad \lambda_2 = \frac{\mu(p^2-q^2)}{(p^2+q^2)^2} = \mu_1\cos\delta,$$

$$\chi_1 = ps-s_1, \quad \chi_2 = qs-s_2, \quad \Phi = \frac{1-\cos\chi_1\cosh\chi_2}{\sin\chi_1\sinh\chi_2}. \tag{27}$$

Then we obtain easily the real form of Vanstone's solution

$$e^{-D} = \frac{1}{y\varrho} = \mu_1(\cosh\chi_2 - \cos\chi_1),$$

$$\tan C = \frac{\beta - cf}{f+c\beta} = \frac{\Phi - \tan\delta}{1+\Phi\tan\delta} \tag{28}$$

from which (with the aid of the second of equations (22)) the components of $g_{\mu\nu}$ can be calculated directly.

We still have to satisfy equation (24). An elementary calculation using (27) and (28) shows that it reduces to an identity providing

$$p^2 = q^2 - 1. \tag{29}$$

We can now proceed to discuss the several cases of the solution of the metric equation (4).

6. $B' = 0,\ w \neq 0$

Because of restrictions imposed in Section 4, it is convenient to revert to equations (20) and (22). We have, if $f \neq 0$, $f \propto \beta$, and from the first of equations (17)

$$\beta = \frac{b_0}{a_0}\,\varrho. \tag{30}$$

Then

$$\Gamma^1_{22} = -\frac{\beta\varrho'}{2\alpha\varrho}, \qquad \Gamma^1_{23} = \tfrac{1}{2}k_0\frac{\varrho'}{\alpha}, \tag{31}$$

where

$$k_0^2 = 1 - \frac{b_0^2}{a_0^2} \qquad (f = \pm k_0\varrho).$$

From (30) we have two cases.
(*i*) $f \neq 0$ implies $k_0 \neq 0$ and the field equations become

$$\left(\frac{\beta\varrho'}{2\alpha\varrho}\right)' + \tfrac{1}{2}(\ln(\alpha y))'\left(\frac{\beta\varrho'}{2\alpha\varrho}\right) - 1 = 0,$$

$$\left(\frac{\varrho'}{\alpha}\right)' + \tfrac{1}{2}(\ln \alpha y)'\frac{\varrho'}{\alpha} + 2\frac{c}{k_0} = 0,$$

$$2A'' - A'^2 + A'\left(\ln\frac{x}{y}\right)' = 0. \tag{32}$$

The last of these equations gives

$$\frac{A'^2 x}{\ln y} = \kappa^2,\ \text{a constant,} \tag{33}$$

whence, because of $\alpha = p^2y'^2/\lambda^2 y$, $y' = \pm\lambda p'/\kappa\varrho^{3/2}$, which is incompatible with (18). Therefore either $\lambda = 0$ or $f = 0$. If $\lambda = 0$, $y =$ constant and disappears from equations (32), but from the second of equations (17) ϱ also is a constant, and this is incompatible with the first of equations (32). Hence we conclude that the condition $B' = 0$ implies $f = 0$ (so that in particular, we cannot have $B' = w = 0$ or the theory collapses into General Relativity).
(*ii*) $f = 0$. This case was solved by Papapetrou but without assuming initially that B is constant as well ($= \pi/2$). The second of equations (32) now implies that

$$c = 0, \tag{34}$$

so that we revert to strong field equations ($R_{\mu\nu} = 0$). We can now put, without loss of generality,

$$a_{11} = -\alpha, \quad a_{22} = -\varrho, \quad \beta = \varrho, \quad a_{44} = \frac{y}{y_0},$$

together with, from equation (33),

$$y' = \frac{\lambda}{\kappa}\varrho^{-3/2}\varrho' \quad \text{or} \quad y = y_1 - \lambda_1 \varrho^{-1/2}, \quad \lambda_0 = \frac{2\lambda}{\kappa}. \tag{35}$$

The only field equation we must still satisfy is

$$\left(-\frac{\varrho'}{2\alpha}\right)' + \tfrac{1}{2}(\ln \alpha y)'\left(-\frac{\varrho}{2\alpha}\right) + 1 = 0,$$

or, since $\dfrac{\varrho'}{\alpha} = \kappa^2 \dfrac{\varrho y}{\varrho'}$,

$$\left(\frac{\varrho y}{\varrho'}\right)' + \tfrac{1}{2}\frac{\varrho y}{\varrho'}\left(\ln \frac{\varrho'^2}{\varrho}\right)' - \frac{2}{\kappa^2} = 0.$$

This is identical with (35) if $y_1 = 4/\kappa^2$. When $\varrho = r^2$, we obtain Papapetrou's solution

$$\alpha = \left(1 - \frac{\lambda_0 \kappa^2}{4r}\right)^{-1}, \quad \sigma = \frac{4}{\kappa^2}\left(1 - \frac{\lambda_0 \kappa^2}{4r}\right)\left(1 + \frac{k^2}{r^4}\right),$$

$$\beta = r^2, \quad w = \pm \frac{2k}{\kappa r^2}. \tag{36}$$

7. $w = 0$

When w is put equal to zero, the equations $\Gamma_\mu = 0$ are identically satisfied and Tonnelat's solution gives

$$\underline{\Gamma^4_{14}} = \tfrac{1}{2}\frac{\sigma'}{\sigma}, \quad \Gamma^1_{44} = \tfrac{1}{2}\frac{\sigma'}{\alpha}.$$

Thus, from (4′)

$$a_{11} = a_0 \alpha, \quad a_{22} = b_0 \varrho \quad \text{and} \quad a_{44} = \sigma_0 \sigma, \tag{37}$$

σ_0 being a new constant. Also, the algebraic conditions (17) become

$$\tfrac{1}{2} b_0 \varrho' + a_0 \alpha \Gamma^1_{22} = 0, \tag{38}$$

whence

$$\Gamma^1_{22} = -\frac{b_0}{2a_0}\frac{\varrho'}{\alpha}, \tag{39}$$

and

$$\tfrac{1}{2}\,\sigma'(\sigma_0+a_0) = 0. \tag{40}$$

Therefore, either $\sigma' = 0$ or $\sigma_0 = -a_0$.

Now if σ is a constant, a lengthy but elementary calculation shows that the field equations (20) are incompatible with the first of the equations (22). Hence

$$\sigma_0 = -a_0, \tag{41}$$

and we can put without loss of generality $a_0 = b_0(= -1)$. We then have $\Gamma^1_{22} = -\varrho'/2\alpha$ and recalling the definition (11) we easily find

$$\sin B = 1-\frac{\varrho}{\varrho_0}, \qquad \cos B = \frac{\varrho}{\varrho_0}\sqrt{2\,\frac{\varrho_0}{\varrho}-1} = \frac{f}{\varrho},$$

$$B' = -\frac{\varrho'}{\varrho\sqrt{2\,\frac{\varrho_0}{\varrho}-1}}, \qquad \beta = \varrho\left(1-\frac{\varrho}{\varrho_0}\right),$$

and

$$\Gamma^1_{\check{2}3} = \frac{\varrho'}{2\alpha\sqrt{2\,\frac{\varrho_0}{\varrho}-1}},$$

where ϱ_0 is a constant. We also have $y = \sigma$, so that relations (22) give

$$\left(\ln\frac{x}{y}\right)' = 2A'-(\ln\alpha\sigma)' \tag{42}$$

and

$$2A''+A'^2+B'^2-(\ln\alpha\sigma)'A' = 0. \tag{43}$$

The equations (17) become

$$\left(\frac{\varrho'}{\alpha}\right)' + (\ln\alpha\sigma)'\,\frac{\varrho'}{\alpha} + \frac{\varrho'^2}{\alpha(2\varrho_0-\varrho)} - 2 = 0, \tag{44}$$

and

$$\left(\frac{\varrho'}{\alpha\sqrt{2\,\frac{\varrho_0}{\varrho}-1}}\right)' + \tfrac{1}{2}(\ln\alpha\sigma)'\,\frac{\varrho'}{\alpha\sqrt{2\,\frac{\varrho_0}{\varrho}-1}} - \frac{\varrho'^2}{\alpha\varrho\sqrt{2\,\frac{\varrho_0}{\varrho}-1}} + 2c = 0, \tag{45}$$

with

$$\sigma'^2 = \frac{\lambda^2\alpha\sigma}{\varrho^2}. \tag{46}$$

Equation (42) can be integrated at once to give

$$\alpha\sigma = \frac{\varrho'^2}{\mu\varrho(2\varrho_0-\varrho)}, \tag{47}$$

μ being the constant of integration. Let $\nu\varrho_0 = -\lambda/\sqrt{\mu}$. Then, substituting (47) into (46) and integrating we find that

$$\alpha = \frac{\varrho'^2}{\mu\varrho(2\varrho_0-\varrho)\left(\omega+\nu\sqrt{2\frac{\varrho_0}{\varrho}-1}\right)}, \tag{48}$$

and

$$\sigma = \omega+\nu\sqrt{2\frac{\varrho_0}{\varrho}-1}, \tag{49}$$

where ω is another constant. Substituting these results into the remaining equations (44) and (45) we find after a straight forward reduction that they are identically satisfied providing

$$\mu\omega\varrho_0-2 = 0, \tag{50}$$

and

$$\mu\nu\varrho_0-2c = 0. \tag{51}$$

ϱ of course, remains an undetermined function of r, the radial coordinate. If we require that $a_{22} = -r^2$, that is $\varrho = r^2$, then the corresponding solution becomes

$$\alpha = \frac{2}{\left(1-\frac{r^2}{r_0^2}\right)\left(1+c\sqrt{\frac{r_0^2}{r^2}-1}\right)},$$

$$\sigma = \omega\left(1+c\sqrt{\frac{r_0^2}{r^2}-1}\right), \quad f = \frac{4r^4}{r_0^2}\sqrt{\frac{r_0^2}{r^2}-1}, \tag{52}$$

where we put $2\varrho_0 = r_0^2$. Clearly, for real components $g_{\mu\nu}$ we must have

$$r < r_0. \tag{53}$$

8. *Discussion*

It may seem curious that the fundamental difference between Einstein's non-symmetric theory and General Relativity, as described in the first two sections of this work, does not appear to have been hitherto emphasised, or indeed, noticed. Yet, an explanation of this is not hard to find. It is partly due to the fact that in General Relativity one decides a priori, on the particular interpretation of geometry (equivalence, or in other words gravitation) while "physics" and geometry, that is the components $g_{\mu\nu}$ of the symmetric metric tensor, are simultaneously determined by the field equations. The second reason is the undue haste with which one wants to revert to a theory known to be successful (General Relativity) and, in the plethora of proposals for a unified field which have been suggested, to recover Maxwell's equations in a virtually unammended form. Einstein alone seemed to see that this may not be possible. In fact, if the new theory is to be empirically verifiable, it may not be desirable. Finally from the mathematical point of view, the choice of a metric (function or relation) is free within relatively weak restrictions (triangle inequality, etc.). Hence it is difficult to realise that, as far as physics is concerned, it may be necessary to postulate not just the overall structure of geometry (viz. Riemannian space of General Relativity) but a law (such as is contained in equation (4)) from which the form of the metric can be found.

We have seen that in the non-symmetric unified field theory there is an initial bifurcation of physics and geometry. The reason for this in the replacement of the Principle of Equivalence which forced the identification of the gravitational field with the Riemannian metric by the much weaker Hermitian or Transposition Invariance. The only thing assumed a priori is the mathematical representation of the physical fields by the sixteen components of the non-symmetric fundamental tensor. And geometry is purely affine in the sense that only the nonsymmetric affine connection appears at the outset. Both $g_{\mu\nu} = g_{\underline{\mu\nu}} + g_{\underset{\vee}{\mu\nu}}$ and $\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\underline{\mu\nu}} + \Gamma^{\lambda}_{\underset{\vee}{\mu\nu}}$ are determined by the field equations. In this sense the latter constitute a realisation of the program of geometrisation of physics as distinct from a geometrisation achieved through an initial hypothesis as in General Relativity. If now we agree that the geometry of a physical world should be metric, it follows that the question of which quantity should be regarded as the appropriate metric tensor appears wide open.

We have proposed in the present work a solution of this problem. At first sight it may seem that the proposal is just as ad hoc as the postulate that $g_{\underline{\mu\nu}}$ is the metric, is clearly too strong since one would expect the metric properties of the space-time of the unified field theory to depend explicitly on the non-symmetric fields. There is nothing in the postulates of the theory to demand this role for $g_{\underline{\mu\nu}}$. The metrics of Schrödinger and of Wyman satisfy these requirements but they are both very artificial. Ultimately, of course, only an empirical study of the metric properties of the space-time, say in the vicinity of a strong electric charge can decide which choice is right. In the absense of data however, our solution has twofold superiority over earlier proposals.

Its first merit is that it allows us to limit the number of possible solutions of the field equations. Vanstone's solution (Section 5) is perfectly general but because of this it is extremely difficult to interpret physically which must be the aim of the theory. As we have seen, the result of requiring equations (4) to hold is that only solutions (36) and (52) appear to be permissible In particular, we cannot have a static, spherically symmetric solution for which

$$w \neq 0, \quad f \neq 0 \tag{54}$$

simultaneously. But we can go further. Both the solutions (36) and (52) result from the choice

$$a_{22} = -r^2, \tag{55}$$

since otherwise they involve, as does Vanstone's general solution, an arbitrary function of r. However, this is not a severe restriction. It is in fact, nothing else than a definition of the radial coordinate. It is difficult to see how one can obtain a meaningful theory without such a definition. The choice (55) means that the "metric" contains an Euclidean 2-sphere. But an Euclidean sphere is a Riemannian space V_2 (with constant, positive curvature).

What is more important is whether solutions (36) and (52) have themselves any physical meaning. The former clearly does if $w = \underset{\vee}{f}_{14}$ is regarded as the electromagnetic field but this interpretation is ruled out by the theorems on the equations of motion (Klotz and Russell, Ref. [12]). On the other hand, the identification (Ref. [11])

$$f_{\mu\nu} = {}^*g^{\alpha\beta} \underset{+\vee-}{g_{\mu\nu;\alpha\beta}} \tag{56}$$

(or, as I prefer, $f_{\mu\nu} = {}^*g^{\alpha\beta} \underset{0\vee 0}{g_{\mu\nu;\alpha\beta}}$) of the electromagnetic field tensor leads to even greater difficulties. For example, the highest power of r in the expression for f_{14} in the Papapetrou case (36) is -4:

$$f_{14} = \frac{4k}{\kappa r^4}\left(1 + \frac{r_0 k^2}{r^5}\right) + \frac{4k^3}{\kappa r^8}\left(\frac{r_0}{r} - 1\right). \tag{57}$$

Far from being fatal, this difficulty of interpretation enables us to resolve another perplexing question in the foundations of the unified field theory. Gregory and the present author have recently (Refs [18, 19]) raised the question of uniqueness of the interpretation (56). They have shown that to the order of approximation required to derive the Lorentz force one can also have

$$f_{\mu\nu} = \underset{\vee}{R}_{\mu\nu}. \tag{58}$$

We can now conclude from (57) that this is the more appropriate identification of the electromagnetic field. If it is adopted then equations (19) immediately imply

$$f_{23} \propto r^{-2}, \tag{59}$$

so that Coulomb law appears to hold absolutely in the unified field theory.

In addition, we also get

$$f_{14} = \underset{\vee}{R}_{14} = -\underset{\vee}{\Gamma}{}^1_{14} + \underset{\vee}{\Gamma}{}^1_{11}\underset{\vee}{\Gamma}{}^1_{14} + 2\underline{\Gamma}{}^2_{12}\underset{\vee}{\Gamma}{}^2_{24} + \underset{\vee}{\Gamma}{}^1_{14}\underline{\Gamma}{}^4_{14} + 2\underset{\vee}{\Gamma}{}^3_{12}\underline{\Gamma}{}^2_{34} - \underset{\vee}{\Gamma}{}^1_{14}\underline{\Gamma}{}^\sigma_{1\sigma}.$$

In the case when $w = 0$ this component of the electromagnetic field tensor vanishes identically. However, when $f = 0$ instead, it becomes

$$-\frac{4kr_0}{\kappa r^5}. \tag{60}$$

This looks like an octupole field though it is perhaps too early in the theory to conclude this firmly. If this solution were rejected as physically meaningless, it would be necessary to assert that spherically symmetric (at any rate) magnetic monopoles cannot exist. The result would be very satisfying as far as unified field theory is concerned.

Work is at present being carried out to extend the results so far obtained to the case of cylindrical symmetry.

We may add that the case $w \neq 0, f \neq 0$ can be excluded as physically meaningful on the grounds that since it implies $c = 0$, and therefore a solution of strong field equations, it does not give a Lorentz force in the equations of motion (Ref. [12]).

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