CONSERVATION LAWS AND IDENTITIES IN THE NON-SYMMETRIC UNIFIED FIELD THEORY

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The conservation laws and Bianchi-type identities are derived for the generalised, non-symmetric unified field theory. Also four identities reducing to the equation of geodesic deviation in the symmetrised case are derived.

1. Introduction

In a previous article (Ref. [1]) it was proposed that the symmetric metric tensor $a_{\mu\nu}$ in the non-symmetric unified field theory should be determined by the equations

$$a_{\mu\nu,\lambda} - \underline{\Gamma}^{\sigma}_{\mu\lambda} a_{\sigma\nu} - \underline{\Gamma}^{\sigma}_{\nu\lambda} a_{\mu\sigma} = 0, \tag{1}$$

where $\underline{\Gamma}^{\lambda}_{\mu\nu}$ are the symmetric components of the affine connection. It was shown that this proposal severely restricts the range of solutions possible in the theory and that it virtually settles the question of identification of the electromagnetic field tensor. It should be emphasized that the identification

$$f_{\mu\nu} = R_{\underset{\vee}{\mu\nu}}, \tag{2}$$

of the latter, removes all dispute as to whether strong or weak field equations should be adopted. Since the former require

$$R_{\underset{\vee}{\mu\nu}} = 0, \tag{3}$$

they are impossible if we wish to have an electromagnetic field at all.

We mean by the metric tensor the quantity required to measure distance between distinct points of the manifold in the presence of gravitational and electromagnetic fields. It is not necessary to raise and lower tensor indices exclusively with its help since any symmetric, non-singular tensor (of rank 2) will do for this purpose.

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If the identification (2) is admitted then it appears that the only physically sensible solution of the weak field equations in the spherically symmetric case, and with $a_{\mu\nu}$ diagonalised, is when the skew symmetric part $g_{\check{\mu\nu}}$ of the fundamental tensor reduces to just one component, $g_{\check{23}}$ (that is, when $w = g_{\check{14}} = 0$). In order to throw more light on the question whether $g_{\check{14}}$ or $g_{\check{23}}$ should be made to vanish (by (1) they cannot both be non-zero), we shall consider now the conservation laws of the theory. It will be shown that this restricts even further the freedom of choice of the field theory (the version of the field equations, since with the elimination of the strong field equations, all possible theories retaining the Principle of Hermitian, or Transposition Invariance are known to be equivalent, Ref. [2]). These questions are, of course, vital to the physical interpretation of the theory and therefore also to its eventual, experimental confirmation.

2. *Conservation laws*

Let us consider the most general (Transposition Invariant) non-symmetric field theory as outlined by Russell and the present author (Ref. [2]). $U^{\lambda}_{\mu\nu}$ is called an Hermitian variable if, when the affine connection $\Gamma^{\lambda}_{\mu\nu}$ is expressed in terms of it, the Ricci tensor

$$R_{\mu\nu} = -\Gamma^{\sigma}_{\mu\nu,\sigma} + \Gamma^{\sigma}_{\mu\sigma,\nu} + \Gamma^{\sigma}_{\mu\varrho}\Gamma^{\varrho}_{\sigma\nu} - \Gamma^{\sigma}_{\mu\nu}\Gamma^{\varrho}_{\sigma\varrho}, \tag{4}$$

automatically becomes Transposition Invariant with respect to $U^{\lambda}_{\mu\nu}$. The most general Hermitian $U^{\lambda}_{\mu\nu}$ is given by

$$\Gamma^{\lambda}_{\mu\nu} = U^{\lambda}_{\mu\nu} + (2\alpha_1 + \tfrac{1}{3})\delta^{\lambda}_{\nu}U^{\sigma}_{\underline{\mu\sigma}} - \tfrac{1}{3}\,\delta^{\lambda}_{\nu}U_{\mu} - (3\alpha_1+1)\delta^{\lambda}_{\mu}U^{\sigma}_{\underline{\nu\sigma}} + (3\alpha_1+2\alpha_2+1)\delta^{\lambda}_{\mu}U_{\nu}, \tag{5}$$

where

$$U_{\mu} \equiv U^{\sigma}_{\check{\mu\sigma}}, \tag{6}$$

and α_1 and α_2 are numerical parameters determining a particular "version" of the theory. Russell and the present author show that a variation in α_2 leads to the conservation law

$$(\mathfrak{g}^{\check{\mu\nu}}\Gamma_{\mu})_{,\nu} = 0. \tag{7}$$

Since the generalised theory requires that

$$\mathfrak{g}^{\check{\mu\nu}}{}_{,\nu} \equiv 0,$$

it follows that

$$\mathfrak{g}^{\check{\mu\nu}}(\Gamma_{\mu,\nu} - \Gamma_{\nu,\mu}) = 0.$$

Now by the identification (2) this is equivalent to

$$\mathfrak{g}^{\check{\mu\nu}}f_{\mu\nu} = 0. \tag{8}$$

In view of the conclusion of the article cited (Ref. [1]), however, such a conservation relation is impossible. Thus we deduce that α_2 cannot be varied and therefore, unless it is put equal to zero (as for example, in the Einstein-Kaufman theory), it must be regarded as a universal constant. Its precise nature cannot be at present determined.

From (5) we easily deduce the transformation law of $U^{\lambda}_{\mu\nu}$, in the form

$$U'^{\lambda}_{\mu\nu} = \frac{\partial x'^{\lambda}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x'^{\mu}} \frac{\partial x^{\gamma}}{\partial x'^{\nu}} U^{\alpha}_{\beta\gamma} + \frac{\partial x'^{\lambda}}{\partial x^{\alpha}} \frac{\partial^2 x^{\alpha}}{\partial x'^{\mu} \partial x'^{\nu}}$$

$$+\mathscr{D}\left[P\delta^{\lambda}_{\nu} \frac{\partial x'^{\sigma}}{\partial x^{\alpha}} \frac{\partial^2 x^{\alpha}}{\partial x'^{\sigma} \partial x'^{\mu}} + Q\delta^{\lambda}_{\mu} \frac{\partial x'^{\sigma}}{\partial x^{\alpha}} \frac{\partial^2 x^{\alpha}}{\partial x'^{\sigma} \partial x'^{\nu}}\right], \tag{9}$$

where

$$P = -9\alpha_1{}^2 - 6\alpha_1\alpha_2 - \alpha_1 - \alpha_2 + 1,$$

$$Q = -(9\alpha_1{}^2 + 6\alpha_1\alpha_2 + 6\alpha_1 + \alpha_2 + 1),$$

$$\mathscr{D}^{-1} = (5\alpha_1 + \tfrac{4}{3})(9\alpha_1 + 6\alpha_2 + 2) \neq 0. \tag{10}$$

Following the method of Weyl and Einstein we consider an infinitesimal co-ordinate transformation

$$x'^{\lambda} = x^{\lambda} + \varepsilon\xi^{\lambda}(x), \qquad \varepsilon^2 \ll \varepsilon, \tag{11}$$

which actually represents a mapping between neighbouring regions of the manifold. Because of this, we get

$$\delta U^{\lambda}_{\mu\nu} = U^{\alpha}_{\mu\nu}\xi^{\lambda}{}_{,\alpha} - U^{\lambda}_{\alpha\nu}\xi^{\alpha}{}_{,\mu} - U^{\lambda}_{\mu\alpha}\xi^{\alpha}{}_{,\nu} - \xi^{\lambda}{}_{,\mu\nu}$$

$$-\mathscr{D}(P\delta^{\lambda}_{\nu}\xi^{\sigma}{}_{,\mu\sigma} + Q\delta^{\lambda}_{\mu}\xi^{\sigma}{}_{,\nu\sigma}) - U^{\lambda}_{\mu\nu,\sigma}\xi^{\sigma}, \tag{12}$$

$$\delta\mathfrak{g}^{\mu\nu} = \mathfrak{g}^{\sigma\nu}\xi^{\mu}{}_{,\sigma} + \mathfrak{g}^{\mu\sigma}\xi^{\nu}{}_{,\sigma} - \mathfrak{g}^{\mu\nu}\xi^{\sigma}{}_{,\sigma} - \mathfrak{g}^{\mu\nu}{}_{,\sigma}\xi^{\sigma}, \tag{13}$$

to the first order in ε. The field equations of the theory are obtained from the (Transposition Invariant) "Action Principle"

$$\delta \int \mathscr{L} d\Omega = \int (R_{\mu\nu}\delta^{\mu\nu} + \mathscr{N}^{\mu\nu}_{\lambda}\delta U^{\lambda}_{\mu\nu}) d\Omega = 0, \tag{14}$$

where the "Lagrangian"

$$\mathscr{L} = \mathfrak{g}^{\mu\nu}R_{\mu\nu}, \tag{15}$$

and the Ricci tensor $R_{\mu\nu}$ is expressed in terms of $U^{\lambda}_{\mu\nu}$ (e. g. equation (9) of Ref. [2]).

Inserting expressions (12) and (13) into (14) we immediately derive equations analogous to the Branchi identities of General Relativity:

$$(R_{\lambda\mu}\mathfrak{g}^{\nu\mu} + R_{\mu\lambda}\mathfrak{g}^{\mu\nu} + \mathscr{N}^{\mu\alpha}_{\lambda}U^{\nu}_{\mu\alpha} - \mathscr{N}^{\nu\mu}_{\alpha}U^{\alpha}_{\lambda\mu} - \mathscr{N}^{\mu\nu}_{\alpha}U^{\alpha}_{\mu\lambda}$$

$$+\mathscr{N}^{\mu\nu}_{\lambda,\mu} + \mathscr{D}P\mathscr{N}^{\nu\alpha}_{\alpha,\lambda} + \mathscr{D}Q\mathscr{N}^{\alpha\nu}_{\alpha,\lambda})_{,\nu} - R_{\mu\nu,\lambda}\mathfrak{g}^{\mu\nu}$$

$$+\mathscr{N}^{\mu\nu}_{\alpha}U^{\alpha}_{\mu\nu,\lambda} = 0. \tag{16}$$

These ensure freedom of choice of the co-ordinate system. We can also derive a conservation law as follows. First, without integration by parts, we obtain

$$\delta\mathscr{L} = (-\mathfrak{g}^{\mu\nu}\delta U^{\sigma}_{\mu\nu}+(1+3\alpha_1)(\mathfrak{g}^{\mu\sigma}\delta U^{\nu}_{\mu\nu}+\mathfrak{g}^{\sigma\mu}\delta U^{\nu}_{\nu\mu})+4\alpha_2\mathfrak{g}^{\mu\sigma}_{\vee}\delta U_{\mu}))_{,\sigma}. \tag{17}$$

Hence, if we specialise transformation (11) by making ξ^{λ} independent of the co-ordinates, and require

$$\delta\mathscr{L} = 0, \tag{18}$$

then

$$\mathfrak{T}^{\mu}_{\nu,\mu} = 0, \tag{19}$$

where

$$\mathfrak{T}^{\mu}_{\nu} = \mathfrak{g}^{\alpha\beta}U^{\mu}_{\alpha\beta,\nu}-(1+3\alpha_1)(\mathfrak{g}^{\alpha\mu}U^{\beta}_{\alpha\beta,\nu}+\mathfrak{g}^{\mu\alpha}U^{\beta}_{\beta\alpha,\nu})-4\alpha_2\mathfrak{g}^{\alpha\mu}_{\vee}U_{\alpha,\nu}. \tag{20}$$

When $\alpha_1 = -\frac{1}{3}$, $\alpha_2 = 0$, this reduces to the Einstein–Kaufman expression

$$\mathfrak{T}^{\mu}_{\nu} = \mathfrak{g}^{\alpha\beta}U^{\mu}_{\alpha\beta,\nu}. \tag{21}$$

Whether $\mathfrak{T}^{\mu}_{\nu}$ can be regarded as energy-momentum is questionable for several reasons. First, and perhaps most important, is that whatever symmetric tensor we use to raise/lower the indices, $\mathfrak{T}^{\mu\nu}$ is not symmetric in general (and its symmetric part is not conserved on its own). It is uncertain at present, what quantity of the form

$$\tau^{\mu\nu\varrho}{}_{,\varrho}, \qquad \tau^{\mu\nu\varrho} = -\tau^{\mu\varrho\nu}, \tag{22}$$

(skew in the last pair of indices so that it may be identically "conserved") should be added to it to overcome this difficulty. Perhaps, once a suitable τ is found, we shall automatically solve the problem (pointed out in Ref. [2]) that $\mathfrak{T}^{\mu\nu}$ is a tensor density only for linear transformations of co-ordinates.

Furthermore, both the "energy-momentum" tensor of Einstein and Kaufman (Ref. [3]) and our $\mathfrak{T}^{\mu}_{\nu}$ are such that for a spherically symmetric, static solution of the field equations,

$$\mathfrak{T}^{4}_{4} = 0, \tag{23}$$

unless $w = 0$. This is strange since we associate this component with energy density. Einstein interprets the above result as implying that a stationary, non-singular field cannot represent a non-zero mass. It seems however, premature to conclude anything until the former questions are resolved. In any case, it is not quite clear what is meant by an energy-momentum tensor in a theory which admits only two physical fields (or, rather, applies only to them). Both gravitational and electromagnetism, the two macroscopic fields represented by the fundamental tensor $g_{\mu\nu}$, are geometrised by our interpretation of the field equations (Ref. [1]). Hence solutions of the latter correspond to the solutions of the field equations of General Relativity in empty space regions. And these certainly do not suffer from lack of physical interpretation because their $T^{4}_{4} = 0$.

Finally, let us rewrite expression (20) in terms of the affine connection $\tilde{\Gamma}^{\lambda}_{\mu\nu}$ of the Einstein-Straus theory:

$$\tilde{\Gamma}^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} + \tfrac{2}{3}\,\delta^{\lambda}_{\mu}\Gamma_{\nu}; \qquad \tilde{\Gamma}_{\mu} = 0, \tag{24}$$

with $\Gamma^{\lambda}_{\mu\nu}$ given by (5). A straightforward calculation gives

$$\begin{aligned} \mathfrak{T}^{\mu}_{\nu} &= \mathfrak{g}^{\alpha\beta}\Gamma^{\mu}_{\alpha\beta,\nu} - \mathfrak{g}^{\alpha\mu}\Gamma^{\beta}_{\alpha\beta,\nu}, \\ &= \mathfrak{g}^{\alpha\beta}\tilde{\Gamma}^{\mu}_{\alpha\beta,\nu} - \mathfrak{g}^{\alpha\mu}\tilde{\Gamma}^{\beta}_{\underline{\alpha\beta},\nu} - \tfrac{4}{3}\,\mathfrak{g}^{\mu\alpha}_{\vee}\Gamma_{\alpha,\nu}, \end{aligned} \tag{25}$$

but, of course, Γ_{μ} cannot be eliminated.

3. "*Geodesic deviation*"

We conclude this note by recording certain identities which reduce to the equation of geodesic deviation in the general relativistic case. In the generalised structure of the unified field theory it is uncertain as to what is meant by a geodesic, or rather, of which space-time is a particular curve a geodesic. Hence it would be incorrect to speak of geodesic deviation of the unified field theory. Nevertheless the formulae are very suggestive and may throw some light on what should be interpreted as force in the unified field.

We start as usual with a two parameter family of surfaces given by

$$x^{\mu} = x^{\mu}(u, v), \tag{26}$$

and define two vectors

$$t^{\mu} = \frac{\partial x^{\mu}}{\partial u}, \qquad n^{\mu} = \frac{\partial x^{\mu}}{\partial v}. \tag{27}$$

Let us also define the operators

$$\overset{+}{D}_{u} n^{\mu} = \frac{\partial n^{\mu}}{\partial u} + \Gamma^{\mu}_{\alpha\beta} n^{\alpha} t^{\beta}, \qquad \overset{-}{D}_{u} n^{\mu} = \frac{\partial n^{\mu}}{\partial u} + \Gamma^{\mu}_{\beta\alpha} n^{\alpha} t^{\beta}, \tag{28}$$

and the vector

$$Q^{\mu} = \frac{\partial t^{\mu}}{\partial u} + \Gamma^{\mu}_{\alpha\beta} t^{\alpha} t^{\beta} = \frac{\partial t^{\mu}}{\partial u} + \Gamma^{\mu}_{\underline{\alpha\beta}} t^{\alpha} t^{\beta}. \tag{29}$$

It is convenient to define two further operators:

$$\nabla^{+}_{v} Q^{\mu} = \frac{\partial Q^{\mu}}{\partial v} + \Gamma^{\mu}_{\alpha\beta} Q^{\alpha} n^{\beta}, \qquad \nabla^{-}_{v} Q^{\mu} = \frac{\partial Q^{\mu}}{\partial v} + \Gamma^{\mu}_{\beta\alpha} Q^{\alpha} n^{\beta}. \tag{30}$$

Then, since

$$\frac{\partial t^{\mu}}{\partial v} = \frac{\partial n^{\mu}}{\partial u}, \tag{31}$$

a straightforward calculation shows that

$$\overset{+-}{D}{}^2_u n^\mu = \frac{\partial}{\partial u}\left(\frac{\partial n^\mu}{\partial u}+\Gamma^\mu_{\beta\alpha} n^\alpha t^\beta\right)+\Gamma^\mu_{\varrho\sigma}\left(\frac{\partial n^\varrho}{\partial u}+\Gamma^\varrho_{\beta\alpha} n^\alpha t^\beta\right)t^\sigma,$$

$$= \nabla^+_v Q^\mu + R^\mu_{\alpha\beta\sigma} n^\sigma t^\alpha t^\beta, \tag{32}$$

the Riemann-Christoffel tensor being

$$R^\mu_{\alpha\beta\sigma} = -\Gamma^\mu_{\alpha\beta,\sigma}+\Gamma^\mu_{\alpha\sigma,\beta}+\Gamma^\varrho_{\alpha\sigma}\Gamma^\mu_{\varrho\beta}-\Gamma^\sigma_{\alpha\beta}\Gamma^\mu_{\varrho\sigma}. \tag{33}$$

Clearly only the symmetric part $R^\mu_{\underline{\alpha\beta}\sigma}$ (in α and β) of the latter enters into this and the succeeding formulae.

In exactly the same way we get

$$\overset{++}{D}{}^2_u n^\mu = \nabla^-_v Q^\mu + 2\Gamma^\mu_{\check{\alpha\beta}}\overset{+}{D} n^\alpha t^\beta + [R^\mu_{\alpha\beta\sigma} + \underset{++}{\overset{+}{\Gamma}}{}^\mu_{\sigma\check{\alpha};\beta} + \underset{++}{\overset{+}{\Gamma}}{}^\mu_{\sigma\check{\beta};\alpha}] n^\sigma t^\alpha t^\beta, \tag{34}$$

$$\overset{(+-)}{D}{}^2_u n^\mu = \nabla^-_v Q^\mu + [R^\mu_{\alpha\beta\sigma} + \underset{++}{\overset{+}{\Gamma}}{}^\mu_{\sigma\check{\alpha};\beta}\underset{++}{\overset{+}{\Gamma}}{}^\mu_{\sigma\check{\beta}\alpha}] n^\sigma t^\alpha t^\beta, \tag{35}$$

and

$$\overset{--}{D}{}^2_u n^\mu = \nabla^+_v Q^\mu + 2\Gamma^\mu_{\check{\alpha\beta}}\overset{+}{D} n^\beta t^\alpha$$

$$+[R^\mu_{\alpha\beta\sigma} + 2\Gamma^\varrho_{\check{\beta\sigma}}\Gamma^\mu_{\check{\alpha\varrho}} + 2\Gamma^\varrho_{\check{\alpha\sigma}}\Gamma^\mu_{\check{\beta\varrho}}] n^\sigma t^\alpha t^\beta. \tag{36}$$

The four expressions (32), (34), (35) and (36) differ from each other, as indeed they must, by a tensor. It is curious how, in our notation, the first of these turns out to be the simplest. It is difficult at the present stage of the theory to give an exact physical interpretation to these results. They leave feeling that there should be some meaning to these formulae.

REFERENCES

[1] A. H. Klotz, *Acta Phys. Pol.* **B9**, 573 (1978).
[2] A. H. Klotz, G. K. Russell, *Acta Phys. Pol.* **134**, 579 (1973).
[3] A. Einstein, *Meaning of Relativity*, Appendix II, London 1956.