THE RELATIONSHIP BETWEEN THE EINSTEIN-STRAUSS AND THE EINSTEIN-KAUFMAN UNIFIED FIELD THEORIES

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The results of Klotz and Russell on the equivalence of the Einstein-Strauss and the Einstein-Kaufman unified field theories are extended to include general Einstein-Kaufman theories.

1. Introduction

The Einstein–Strauss theory [2] was originally derived using the Hamiltonian

$$\mathscr{H} = \mathfrak{g}^{\mu\nu}P_{\mu\nu}+\mathscr{A}^{\mu}\Gamma_{\mu}+b_{\mu}\mathfrak{g}^{\underset{\vee}{\mu\nu}}{}_{,\nu},$$

where $\mathscr{A}^{\mu}$ and b_{μ} are Lagrangian multipliers and $P_{\mu\nu}$ is the Hermitian symmetric tensor

$$P_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha}-\tfrac{1}{2}(\Gamma^{\alpha}_{\underline{\mu\alpha},\nu}+\Gamma^{\alpha}_{\underline{\alpha\nu},\mu})-\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\alpha\nu}+\Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\underline{\alpha\beta}}.$$

A line and a hook under two indices denote the symmetric and the skew-symmetric part of a given quantity, respectively.

The resulting field equations are

$$g_{\underset{+}{\mu}\underset{-}{\nu};\lambda} \equiv g_{\mu\nu,\lambda}-\Gamma^{\sigma}_{\mu\lambda}g_{\sigma\nu}-\Gamma^{\sigma}_{\lambda\nu}g_{\mu\sigma} = 0,$$

$$\Gamma_{\mu} \equiv \Gamma^{\alpha}_{\underset{\vee}{\mu\alpha}} = 0,$$

$$P_{\underline{\mu\nu}} = 0,$$

$$P_{\underset{\vee}{\mu\nu},\lambda} = P_{\mu\nu,\lambda}+P_{\lambda\mu,\nu}+P_{\nu\lambda,\mu} = 0.$$

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This set is also equivalent to the system

$$g_{\underset{+}{\mu}\underset{-}{\nu};\lambda} = 0, \tag{1}$$

$$\Gamma_\mu = 0, \tag{2}$$

$$R_{\underline{\mu\nu}} = 0, \tag{3}$$

$$R_{\underset{\vee}{\mu\nu},\lambda} = 0, \tag{4}$$

where $R_{\mu\nu}$ is the generalized Ricci-tensor

$$R_{\mu\nu} = \Gamma^{\sigma}_{\mu\sigma,\nu} - \Gamma^{\sigma}_{\mu\nu,\sigma} + \Gamma^{\varrho}_{\mu\sigma}\Gamma^{\sigma}_{\varrho\nu} - \Gamma^{\varrho}_{\mu\nu}\Gamma^{\sigma}_{\varrho\sigma}.$$

Einstein was critical of the derivation of the Einstein–Strauss (E.S.) equations and looked for a method of extending General Relativity without introducing artificial devices. In 1955, Einstein and Kaufman [1] proposed a new theory which was strongly motivated by the idea of Hermitian invariance, although not necessarily with respect to $\Gamma^{\lambda}_{\mu\nu}$ (see Klotz and Russell [4] for a definition of general Hermitian invariance). The Hamiltonian used in the new theory was

$$\mathcal{H} = \mathfrak{g}^{\mu\nu}R_{\mu\nu}$$

and variations were taken not with respect to $g_{\mu\nu}$ and $\Gamma^{\lambda}_{\mu\nu}$, but with respect to the variables $g_{\mu\nu}$ and $U^{\lambda}_{\mu\nu}$, where

$$U^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\alpha}_{\mu\alpha}\delta^{\lambda}_{\nu}.$$

The variables $U^{\lambda}_{\mu\nu}$ were chosen to make $R_{\mu\nu}$ Hermitian symmetric as can easily be seen from the expression of $R_{\mu\nu}$ in terms of $U^{\lambda}_{\mu\nu}$

$$R_{\mu\nu} = U^{\alpha}_{\mu\nu,\alpha} - U^{\alpha}_{\mu\beta}U^{\beta}_{\alpha\nu} + \tfrac{1}{3}\,U^{\alpha}_{\mu\alpha}U^{\beta}_{\beta\nu}.$$

The Einstein–Kaufman (E.K.) field equations are

$$\mathcal{N}^{\mu\nu}_{\lambda} = -\mathfrak{g}^{\mu\nu}{}_{,\lambda} - \mathfrak{g}^{\alpha\nu}(U^{\mu}_{\alpha\lambda} - \tfrac{1}{3}\,\delta^{\mu}_{\lambda}U^{\beta}_{\alpha\beta}) - \mathfrak{g}^{\mu\alpha}(U^{\nu}_{\lambda\alpha} - \tfrac{1}{3}\,\delta^{\nu}_{\lambda}U^{\beta}_{\beta\alpha}) = 0$$

and

$$R_{\mu\nu} = 0.$$

$\mathcal{H}$ is also invariant under the transformation

$$\mathfrak{g}^{*\mu\nu} = \mathfrak{g}^{\mu\nu},$$

$$U^{*\lambda}_{\mu\nu} = U^{\lambda}_{\mu\nu} + (\delta^{\mu}_{\nu}\lambda_{,\lambda} - \delta^{\mu}_{\lambda}\lambda_{,\nu}).$$

that is, a "λ-transformation". Infinitesimal λ-transformations produced the additional equations

$$\mathfrak{g}^{\underset{\vee}{\mu\nu}}{}_{,\nu} = 0$$

and

$$(\mathcal{N}^{\cdot\mu\nu}{}_{\mu} - \mathcal{N}^{\nu\mu}{}_{\mu})_{,\nu} = 0.$$

For these reasons, Einstein considered E.K. to be "stronger" and more "natural" than E.S., and hence, preferable.

Although the E.K. theory appears attractive, it has been shown by Klotz [3] that $U^{\lambda}_{\mu\nu}$ is not the only choice of variable to make $R_{\mu\nu}$ Hermitian symmetric. If

$$\Gamma^{\lambda}_{\mu\nu} = V^{\lambda}_{\mu\nu} - \tfrac{1}{3} V^{\sigma}_{\mu\sigma}\delta^{\lambda}_{\nu} - \tfrac{1}{3} V_{\nu}\delta^{\lambda}_{\mu},$$

where

$$V_{\nu} = \tfrac{1}{2}(V^{\sigma}_{\nu\sigma} - V^{\sigma}_{\sigma\nu}),$$

the tensor $R_{\mu\nu}$ is Hermitian-symmetric with respect to $V^{\lambda}_{\mu\nu}$. Thus, the Einstein–Kaufman type of theory is not unique.

2. *Linear Einstein–Kaufman theories*

Klotz and Russell [4] extended Klotz's result and produced a two-parameter (α_1 and α_2) set of theories of the E.K.type. They examined linear, invertible transformations of the form

$$\Gamma^{\lambda}_{\mu\nu} = U^{\lambda}_{\mu\nu} + (2\alpha_1 + \tfrac{1}{3})\delta^{\lambda}_{\nu} U^{\sigma}_{\underline{\mu\sigma}} - \tfrac{1}{3} U^{\lambda}_{\nu} U_{\mu} - (3\alpha_1 + 1)\delta^{\lambda}_{\mu} U^{\sigma}_{\underline{\sigma\nu}}$$
$$+ (3\alpha_1 + 2\alpha_2 + 1)\delta^{\lambda}_{\mu} U_{\nu},$$

such that $R_{\mu\nu}(U)$ is Hermitian symmetric. Consequently, the field equations

$$R_{\mu\nu} = 0, \tag{5}$$

$$\mathcal{U}^{\mu\nu}{}_{\lambda} = 0, \tag{6}$$

are Hermitian with respect to $U^{\lambda}_{\mu\nu}$.

These theories, based upon linear substitutions and Hermitian symmetry, will be called linear Einstein–Kaufman theories.

Klotz and Russell also showed that the linear E.K. equations can be expressed in terms of $\Gamma^{\lambda}_{\mu\nu}$ as

$$g_{\underset{+}{\mu}\underset{-}{\nu};\lambda} - \tfrac{2}{3}\Gamma_{\lambda} g_{\mu\nu} - \tfrac{2}{3}\Gamma_{\nu} g_{\mu\lambda} = 0, \tag{7}$$

$$\mathfrak{g}^{\underset{\vee}{\mu\nu}}{}_{,\nu} = 0, \tag{8}$$

$$R_{\mu\nu} = 0, \tag{9}$$

independently of α_1 and α_2. Further, equations (7), (8) and (9) can be transformed to the set

$$g_{\underset{+}{\mu}\underset{-}{\nu};\lambda}(\tilde{\Gamma}) = 0,$$

$$\tilde{\Gamma}_{\mu} = 0,$$

$$R_{\underline{\mu\nu}}(\tilde{\Gamma}) = 0,$$

$$R_{\underset{\cdot\vee\cdot\;\cdot}{\mu\nu,\lambda}}(\tilde{\Gamma}) = 0,$$

for some affine-connection $\tilde{\Gamma}$, and vice-versa. Thus, the linearised E.K. theories and E.S. are interchangeable.

It is possible to generalise this result to any theory based on the Hamiltonian $\mathscr{H} = \mathfrak{g}^{\mu\nu}R_{\mu\nu}$ where $R_{\mu\nu} = R_{\mu\nu}(U^{\lambda}_{\mu\nu})$ and $U^{\lambda}_{\mu\nu}$ is some invertible function of $\Gamma^{\lambda}_{\mu\nu}$.

3. *Lemma*

The field equations obtained by the variational principle

$$\delta \int_V \mathscr{L}(\mathfrak{g}^{\mu\nu}, U^{\lambda}_{\mu\nu}, U^{\lambda}_{\mu\nu,\alpha})d^n_d x, \qquad d^n x = \prod_{i=1}^{n} dx^i,$$

with $\mathfrak{g}^{\mu\nu}$ and $U^{\lambda}_{\mu\nu}$ as the (independent) variational parameters are given by the Euler-Lagrange equations

$$\frac{\partial \mathscr{L}}{\partial \mathfrak{g}^{\mu\nu}} = 0, \tag{10}$$

$$\frac{\partial \mathscr{L}}{\partial U^{\lambda}_{\mu\nu}} - \frac{\partial}{\partial x^{\alpha}}\left(\frac{\partial \mathscr{L}}{\partial U^{\lambda}_{\mu\nu,\alpha}}\right) = 0. \tag{11}$$

If an invertible change of dependent variables is made,

$$V^{\lambda}_{\mu\nu} = V^{\lambda}_{\mu\nu}(U^{\lambda}_{\mu\nu})$$

and Jacobian $(V, U) \equiv \det\left(\frac{\partial V^{\lambda}_{\mu\nu}}{\partial U^{\gamma}_{\alpha\beta}}\right) \neq 0$, then variation of

$$\int_V \mathscr{L}(\mathfrak{g}^{\mu\nu}, U^{\lambda}_{\mu\nu}, U^{\lambda}_{\mu\nu,\alpha})d^n x = \int_V \mathscr{H}(\mathfrak{g}^{\mu\nu}, V^{\lambda}_{\mu\nu}, V^{\lambda}_{\mu\nu,\alpha})d^n x$$

gives the equations

$$\frac{\partial \mathfrak{g}}{\partial \mathfrak{g}^{\mu\nu}} = 0, \tag{12}$$

$$\frac{\partial \mathscr{H}}{\partial V^{\lambda}_{\mu\nu}} - \frac{\partial}{\partial x^{\alpha}}\left(\frac{\partial \mathscr{H}}{\partial V^{\lambda}_{\mu\nu,\alpha}}\right) = 0. \tag{13}$$

Now it is possible to show that

$$\frac{\partial \mathscr{L}}{\partial U^{\lambda}_{\mu\nu}} - \frac{\partial}{\partial x^{\alpha}}\left(\frac{\partial \mathscr{L}}{\partial U^{\lambda}_{\mu\nu,\alpha}}\right) = \frac{\partial V^{\gamma}_{\delta\varepsilon}}{\partial U^{\lambda}_{\mu\nu}}\left(\frac{\partial \mathscr{H}}{\partial V^{\gamma}_{\delta\varepsilon}} - \frac{\partial}{\partial x^{\alpha}}\left(\frac{\partial \mathscr{H}}{\partial V^{\gamma}_{\delta\varepsilon,\alpha}}\right)\right).$$

Hence, equations (10) and (11) are equivalent to equations (12) and (13).

4. *General Einstein–Kaufman theories*

This theory can now be applied to the Hamiltonian $\mathscr{H} = \mathfrak{g}^{\mu\nu}R_{\mu\nu}$. We will call any theory of the type

$$R_{\mu\nu} = R_{\mu\nu}(U^{\lambda}_{\mu\nu}, U^{\lambda}_{\mu\nu,\alpha}),$$

where $U^{\lambda}_{\mu\nu}$ is an invertible function of $\Gamma^{\lambda}_{\mu\nu}$, a general Einstein–Kaufman theory ($R_{\mu\nu}$ will not necessarily be Hermitian-symmetric with respect to $U^{\lambda}_{\mu\nu}$ although this case is included in the definition).

Lichnerowicz [5] has shown that the variation of

$$\int_V \mathfrak{g}^{\mu\nu}R_{\mu\nu}(\Gamma^{\lambda}_{\mu\nu}, \Gamma^{\lambda}_{\mu\nu,\alpha})d^4x$$

produces the equations

$$g^{\mu\nu}{}_{;\varrho} - (\Gamma^{\sigma}_{\varrho\sigma} - \gamma_{\varrho})g^{\mu\nu} + 2g^{\mu\sigma}\Gamma^{\nu}_{\underset{\vee}{\varrho\sigma}} + \tfrac{2}{3}\,\delta^{\nu}_{\varrho}g^{\mu\sigma}\Gamma^{\alpha}_{\underset{\vee}{\alpha\sigma}} \tag{14}$$

and

$$R_{\mu\nu} = 0, \tag{15}$$

where

$$\gamma_{\varrho} = \tfrac{1}{2}\,\partial_{\varrho} \ln |g|.$$

By using the Schrödinger substitution

$$\Gamma^{*\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} + \tfrac{2}{3}\,\Gamma_{\nu}\sigma^{\lambda}_{\mu},$$

equations (14) and (15) transform to the set

$$g_{\underset{+}{\mu}\,\underset{-}{\nu};\lambda}(\Gamma^*) = 0,$$

$$\Gamma^*_{\mu} = 0,$$

$$R_{\underline{\mu\nu}}(\Gamma^*) = 0,\ \cdot$$

$$R_{\underset{\cdot\vee\cdot}{\mu\nu,\lambda}}(\Gamma^*) = 0,$$

that is, the E. S. field equations. (It is also possible to transform back to equations (14) and (15).)

If we now vary

$$\int_V \mathfrak{g}^{\mu\nu}R_{\mu\nu}(V^{\lambda}_{\mu\nu}, V^{\lambda}_{\mu\nu,\alpha})d^4x,$$

where $V^{\lambda}_{\mu\nu}$ is an invertible function of $\Gamma^{\lambda}_{\mu\nu}$, the Lemma reveals that the system of field equations produced is equivalent to the set (14) and (15), and hence, the E. S. system. Thus the general E. K. theories and E. S. are equivalent in the sense that there is a 1–1 correspondence between their solution sets although the affine connections are not identical.

5. *λ-invariance*

In their work on linear E. K. theories, Klotz and Russell (4) showed that the equation

$$\mathfrak{g}^{\underset{\vee}{\mu\nu}}{}_{,\nu} = 0$$

is a consequence of equation (6). In fact, equation (6) is equivalent to the two equations

$$\mathfrak{g}^{\underset{\vee}{\mu\nu}}{}_{,\nu} = 0$$

and

$$g_{\underset{+}{\mu}\underset{-}{\nu};\lambda} - \tfrac{2}{3}\, g_{\mu\nu}\Gamma_{\lambda} - \tfrac{2}{3}\, g_{\mu\nu}\Gamma_{\lambda} = 0.$$

Since the general E. K. theories can be (invertibly) transformed to the linear E. K. theories,

$$\text{general } V^{\lambda}_{\mu\nu} \overset{\text{invertible}}{\longleftrightarrow} \Gamma^{\lambda}_{\mu\nu} \overset{\text{invertible}}{\longleftrightarrow} \text{linear } U^{\lambda}_{\mu\nu},$$

the equation $\mathfrak{g}^{\underset{\vee}{\mu\nu}}{}_{,\nu} = 0$ is contained within general E. K. theory.

Thus λ-invariance does not contribute any equations that are not already present in the general E. K. theory and so the question of "strength" does not distinguish between the general E. K. and E. S. theories.

6. *Conclusions*

The non-uniqueness of the Einstein-Kaufman theory destroys Einstein's claim that "the U are more natural variables than Γ" and that the Einstein-Kaufman derivation avoids "artificial tricks". Even the property of "strength" fails to clearly distinguish between these theories. Hence the general Einstein-Kaufman theories and the Einstein-Strauss theory are equivalent and neither form is superior to the other.

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