DYNAMICS OF A THIN SHELL IN REISSNER-NORDSTRØM GEOMETRY*

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The radial motion of a thin spherical shell of dust approximated by a singular surface layer in Schwarzschild and Reissner-Nordstrøm geometries is discussed in terms of an effective potential and a development of an event horizon, during collapse of a shell, is investigated.

1. *Introduction*

A spherically symmetric collapse of a cloud of dust surrounding a central black hole can be investigated in two different manners:

(*i*) each particle of the cloud is treated as a test particle moving in a background geometry generated by the central black hole and the inner part of the cloud or

(*ii*) one can represent the cloud by a set of thin shells with finite energies E (and finite electric charges Q in the case of charged dust) and therefore separately solve equations of motion for each shell in the geometry generated by the matter inside it.

In the first approach when a test particle of energy E and charge q falls onto a black hole described by a mass m and a charge e, the change in black hole parameters is $\delta m = E$ and $\delta e = q$. Thus to obtain formulas for final m and e one has to integrate infinitesimal changes over the whole process of collapse. On the other hand, the second method gives such exact formulas immediately.

2. *Dynamics of a thin shell*

The dynamics of a thin spherical shell of dust in Schwarzschild and Reissner-Nordstrøm geometries was investigated by Israel [1] and de la Cruz and Israel [2]. Kuchař [3] and Chase [4] considered more general case of fluid shells. One can find the equation

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of motion approximating a thin shell by a surface layer on which the energy and electric charge densities are singular.

The world lines of particles of the shell form a timelike hypersurface Σ which divides spacetime into two parts. The inner part V_1, in general, is described by the Reissner-Nordstrøm line element (units in which $G = c = 1$ are used)

$$(ds^2)_1 = f_1 dt_1^2 - f_1^{-1} dr^2 - r^2 d\Omega^2, \tag{1}$$

where

$$f_1(r) = 1 - \frac{2m_1}{r} + \frac{e_1^2}{r^2} \tag{2}$$

with m_1 and e_1 being the mass and electric charge of a central black hole. In outer region V_2 the metric is similarly

$$(ds^2)_2 = f_2 dt_2^2 - f_2^{-1} dr^2 - r^2 d\Omega, \tag{3}$$

where

$$f_2(r) = 1 - \frac{2m_2}{r} + \frac{e_2^2}{r^2} \tag{4}$$

with $m_2 = m_1 + E$ and $e_2 = e_1 + Q$. Here E and Q are energy-at-infinity and charge of the shell respectively. The hypersurface Σ is described by $r = R(\tau)$ and has intrinsic metric

$$(ds^2)_\Sigma = d\tau^2 - [R(\tau)]^2 d\Omega^2. \tag{5}$$

The mathematical grounds of how to relate the matter stress-energy tensor and a jump in the energy-momentum tensor of the electromagnetic field to the discontinuity of the extrinsic curvature of Σ are given in [1–3]. The equation of a pure radial motion for the spherical shell of dust (with surface area equal $4\pi R^2$) obtained by de la Cruz and Israel [2] is[1]

$$1 + \left(\frac{dR}{d\tau}\right)^2 = A + \frac{B}{R} + \frac{C}{R^2} \tag{6}$$

with

$$B = m_1 + m_2 - A(e_2^2 - e_1^2)/(m_2 - m_1), \tag{7}$$

$$4C = A(e_2^2 - e_1^2)^2/(m_2 - m_1)^2 - 2(e_1^2 + e_2^2) + (m_2 - m_1)^2/A \tag{8}$$

and the constant $A^{1/2} = (m_2 - m_1)/\mathcal{M}$ being the ratio of a total gravitational energy $E = m_2 - m_1$ to the "nucleonic" mass $\mathcal{M}$ of the shell . The nucleonic mass $\mathcal{M}$ is the sum of the rest masses of all particles forming the shell. The equations (6)÷(8) are valid also for charged fluid shell [4]. In such a case the nucleonic mass $\mathcal{M}$ appearing in definition

[1] Eqs (30)÷(32) from [2].

of A has to be replaced by the total proper mass being the sum of $\mathcal{M}$ and the internal thermal energy. The total proper mass is not conserved ($d\mathcal{M}_{\text{total}} = -pd(4\pi R^2)$ for adiabatic contraction) what means that A, B and C are no longer constant of motion.

The equation (6) can be written in a more convenient form

$$\mathcal{M}^2\left(\frac{dR}{d\tau}\right)^2 = \left(E - \frac{\alpha}{R}\right)^2 - \left(1 - \frac{2m_1}{R} + \frac{e_1^2}{R^2}\right)\mathcal{M}^2 \tag{9}$$

with

$$\alpha = e_1 Q + \frac{Q^2 - \mathcal{M}^2}{2}\,. \tag{10}$$

In the case $\mathcal{M} \to 0$ and $Q \to 0$, what corresponds to setting down $\alpha = e_1 Q$, one gets an equation of radial motion for test particle (with zero angular momentum). By introducing effective potential in a well-known manner (for example [5, 6]) one rewrites equation (9) as

$$\mathcal{M}^2\left(\frac{dR}{d\tau}\right)^2 = [E - U^+(R)]\,[E - U^-(R)] \tag{11}$$

with effective potential

$$U^{\pm}(R) = \frac{\alpha}{R} \pm \left(1 - \frac{2m_1}{R} + \frac{e_1^2}{R^2}\right)^{1/2}\mathcal{M}. \tag{12}$$

The asymptotic formula of $U^+(R)$ at large values of R:

$$U^+(R \to +\infty) = \frac{e_1 Q}{R} + \frac{Q^2}{2R} - \frac{\mathcal{M}^2}{2R} - \frac{m_1\mathcal{M}}{R} + \mathcal{M} \tag{13}$$

agrees with the potential describing similar process in Newtonian theory. Thus the terms $e_1 Q/R$ and $Q^2/2R$ in Eq. (10) can be interpreted as representing the shell-hole and shell--shell electric interactions and the term $-\mathcal{M}^2/2R$ as representing shell-shell gravitational interaction.

In the test particle approach ($\alpha = e_1 Q$) the positive and negative root of the effective potential define the regions of space attainable for particles and antiparticles respectively. The states with energies less than negative root of effective potential, i.e. negative root states, correspond to 4-momentum pointing toward the past (antiparticles) while positive root states correspond to future-pointing 4-momentum (particles).

Here only a part of $U^+(R)$ represents a set of turning points for the shells with different energies and due to the nonlinearity of $U^-(R)$ in Q and $\mathcal{M}$ this root is not a set of turning points for a shell made of antiparticles. In order to examine the structure of $U^+(R)$ we apply the process of energy extraction to the physically well defined shell, e.g. to a shell falling down from infinity. The most effective procedure of energy extraction is to lower the shell in a quasi-static manner (with infinitesimally small radial velocity), then the shell energy for each value of R is infinitesimally greater than $U^+(R)$. Such a procedure is possible, in principle, as long as the tidal gravitational forces are finite, what happens if the radius of the shell is greater than the gravitational radius of configuration, that means when

$$R > r_g \equiv m_2 + (m_2^2 - e_2^2)^{1/2} \tag{14}$$

with $m_2 = m_1 + U^+(R)$. It turns out that this does not happen for

$$R = r_0 \equiv m_1 + (m_1^2 - e_1^2 + \mathcal{M}^2)^{1/2} \tag{15}$$

for which both sides of (14) are equal. It means that even if the most effective procedure of contraction of the shell is realized an event horizon appears outside the shell when it crosses $R = r_0$. Thus the minimal change in black hole mass, when the shell of a charge Q and nucleonic mass $\mathcal{M}$ falls in, is

$$E_0 = U^+(r_0) \equiv (e_1 Q + Q^2/2 + \mathcal{M}^2/2)/r_0. \tag{16}$$

In general case of a shell with energy E an event horizon appears at

$$r_g = m_1 + E + [(m_1 + E)^2 - (e_1 + Q)^2]^{1/2} \geqslant r_0. \tag{17}$$

Although the inequality (14) holds also for

$$m_1 + (m_1^2 - e_1^2)^{1/2} < R < r_0 \tag{18}$$

so there are states representing the shells outside the gravitational radius even for $E < E_0$, they are nonattainable for shell being outside r_0 and they are not interesting from physical point of view.

The formulas (11) and (12), after replacing constant $\mathcal{M}$ by the total proper mass $\mathcal{M}_{\text{total}}$ (R), are valid also for a perfect fluid shell.

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