THE DOPPLER AND GRAVITATIONAL COMPONENTS OF THE COSMOLOGICAL REDSHIFT

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We decompose the cosmological redshift in the standard Friedmann cosmologies into two shifts: a Doppler shift attributable to the recession of the galaxies, and a gravitational shift attributable to the curvature of the universe. For galaxies nearby enough for their recessional motion to be non-relativistic, we interpret our results for the Doppler and gravitational shifts with the aid of Birkhoff's theorem.

1. *Introduction*

It has occasionally been noted (see e.g. Rindler [1] and Weinberg [2]) that the cosmological redshift cannot properly be regarded as a purely Doppler shift, for the light from the distant galaxies travels through the gravitational field of the universe, which exerts some effect on the wavelength. Thus the cosmological redshift is in fact a combination of two effects: a Doppler effect due to the recessional motion of the galaxies, and a gravitational effect due to the curvature of the spacetime. In this paper we calculate the separate contributions of these two effects to the cosmological redshift, and we interpret our results for the case of small velocities of recession in terms of Birkhoff's theorem [3]. We also comment on a related field-theoretic calculation by Westervelt [4] of the shift suffered by a photon traveling through a homogeneous, static distribution of dust.

2. *Definition of Doppler shift and gravitational shift*

Here and in general the notions of "Doppler shift" and "gravitational shift" in general relativity are to a certain extent matters of convention (see Synge [5] for a discussion of this point). We shall define them in the following way. Consider three observers A, A', and B. At the event P_e, B emits a light beam whose frequency he measures to be ν_B. A absorbs

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the light beam at the event P_a, measuring a frequency ν_{BA}. A' is an observer whose world-line passes through P_e, at which moment he is "stationary" (a term we will define below) with respect to A. We define the Doppler shift of B relative to A to be the Doppler shift of B relative to A', and we define the gravitational shift of B relative to A to be the shift of A' relative to A. The Doppler shift in this definition is thus due entirely to the relative motion of A and B (i.e. of A' and B) and not at all to their separation, while the gravitational shift is due entirely to their separation and not at all to their relative motion. In a flat spacetime, the gravitational shift in this definition vanishes, leaving only the usual Doppler shift.

It remains for us to define what we mean by an observer A' who is "stationary" with respect to A at the event P_e. It is perhaps most natural to say that A' is stationary with respect to A at P_e if the four-velocity $u^i_{A'}$ of A' at P_e coincides with the four-velocity u^i_A of A at P_a when $u^i_{A'}$ is parallel-transported to P_a along the light beam joining P_e and P_a. However, in a Friedmann universe another natural definition, which we elect to use because it simplifies the calculations somewhat, suggests itself when the observers A and B

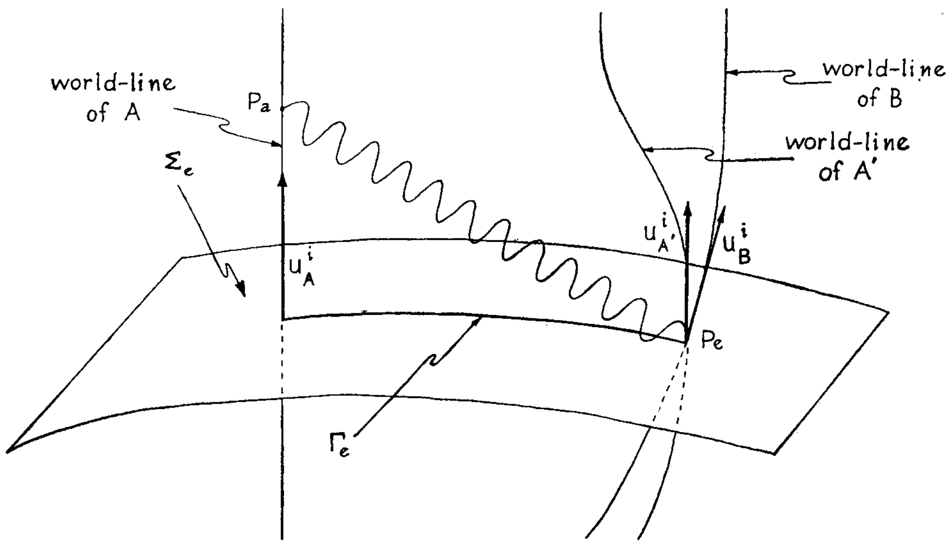

Fig. 1. The geometrical construction for the decomposition of the cosmological shift into its Doppler and gravitational components. The photon is emitted by co-moving observer B at the event P_e and absorbed by co-moving observer A at the revent P_a. A' is an observer who at P_e is stationary with respect to A: A''s four velocity $u^i_{A'}$ is determined by parallel-transporting A's four-velocity u^i_A along the geodesic Γ_e joining A' and A in the three-space Σ_e of constant cosmic time t_e. The cosmological Doppler shift of B relative to A is the shift of B relative to A'. The cosmological gravitational shift of B relative to A is the shift of A' relative to A

are chosen to be co-moving with the cosmological substratum: we shall say that, at the cosmic time t_e of the event P_e, A' is stationary with respect to A if $u^i_{A'}$ coincides with the four-velocity of A when parallel-transported to A along the geodesic Γ_e that joins A' and A in the three-space Σ_e given by $t = t_e$ (see Fig. 1).

Here we note without proof that the two definitions of "stationary" given above, and presumably other equally natural definitions as well, lead to the same result for their corresponding gravitational and Doppler shifts in the case of small velocities of recession.

3. Calculation of the shifts in a Friedmann universe

With our definitions, it is easy immediately to write down the formulae for the Doppler and gravitational shifts in a Friedmann universe. Let $\nu_{BA'}$ be the frequency measured by A' of a light beam emitted by B with frequency ν_B, let $\nu_{A'A}$ be the frequency measured by A of a light beam emitted by A' with frequency $\nu_{A'}$, and let the subscripts e and a denote the events of emission and absorption, respectively. Then the cosmological Doppler shift is given simply by the special relativistic formula

$$\nu_{BA'}/\nu_B = [\gamma_e + (\gamma_e^2 - 1)^{1/2}]^{-1}, \tag{1}$$

where

$$\gamma_e \equiv -(g_{ab} u_{A'}^a u_B^b)_e; \tag{2}$$

and, using the well-known formula [6] for the full cosmological shift $\nu_{BA}/\nu_B = R_e/R_a$ (here $R(t)$ is the scale size of the universe at cosmic time t), the cosmological gravitational shift is given by

$$\nu_{A'A}/\nu_{A'} = (R_e/R_a)\nu_B/\nu_{BA'}. \tag{3}$$

The factor $\nu_B/\nu_{BA'}$ in Eq. (3) removes the Doppler component from the full shift.

We shall now compute γ_e. In the standard coordinates, the metric is given by the fundamental form [7]

$$\varepsilon ds^2 = -dt^2 + R^2(t)\,[(1-k\xi^2)^{-1} d\xi^2 + \xi^2 d\theta^2 + \xi^2 \sin^2\theta d\phi^2], \tag{4}$$

where $\varepsilon = -1, 0$, or 1 if the interval dx^i is timelike, null, or spacelike, respectively, and $k = -1, 0$, or 1 if the geometry of Σ is hyperbolic, Euclidean, or spherical, respectively. In these coordinates, the four-velocity of the comoving observer B is $u_B^i = (1, 0, 0, 0)$, so, from Eqs (2) and (4),

$$\gamma_e = (u_{A'}^0)_e. \tag{5}$$

To find $u_{A'}^0$ we must solve the equation for parallel transport

$$du_{A'}^0 + \Gamma_{ab}^0 u_{A'}^a dx^b = 0 \tag{6}$$

along Γ_e. For Γ_e, $dx^i = (0, d\xi, 0, 0)$, so Eq. (6) becomes

$$du_{A'}^0/d\xi + \Gamma_{a\xi}^0 u_{A'}^a = 0. \tag{7}$$

From Eq. (4) we find that the only non-zero $\Gamma_{a\xi}^0$ is $\Gamma_{\xi\xi}^0 = HR^2(1-k\xi^2)^{-1}$, where $H \equiv \dot{R}/R$ is Hubble's constant, so Eq. (7) becomes

$$du_{A'}^0/d\xi + H_e R_e^2 (1-k\xi^2)^{-1} u_{A'}^\xi = 0. \tag{8}$$

Using the relation $-1 = -(u^0_{A'})^2 + R_e^2(u^\xi_{A'})^2(1-k\xi^2)^{-1}$ for the components of the unit vector $u^i_{A'}$, we may express $u^\xi_{A'}$ in terms of $u^0_{A'}$ in Eq. (8):

$$du^0_{A'}/d\xi - H_e R_e[(u^0_{A'})^2-1]^{1/2}(1-k\xi^2)^{-1/2} = 0. \tag{9}$$

Since $u^0_{A'} = u^0_A = 1$ at the origin $\xi = 0$ where we take A to be located, the solution to Eq. (9) is

$$u^0_{A'} = \cosh H_e r, \tag{10}$$

where

$$r \equiv R_e \int_0^\xi (1-k\xi^2)^{-1/2} d\xi \tag{11}$$

is the proper distance from the origin out to ξ as measured along Γ_e.

From Eqs (1), (5), and (10), then, the cosmological Doppler shift is given by

$$\frac{\nu_{BA'}}{\nu_B} = \exp(-H_e r_e), \tag{12}$$

where r_e is the proper distance from A to B at the time of the emission at B. The cosmological gravitational shift, from Eq. (3), is given by

$$\frac{\nu_{A'A}}{\nu_{A'}} = (R_e/R_a)\exp H_e r_e. \tag{13}$$

It is easy to see that the cosmological Doppler shift in Eq. (12) has a plausible form. For in the Friedmann cosmologies the velocity of recession of a point in the substratum a distance r from the origin is $v \equiv dr/dt = Hr$, so that Eq. (12) may be rewritten

$$\frac{\nu_{BA'}}{\nu_B} = \exp(-v_e) = 1 - v_e + \tfrac{1}{2}v_e^2 - \tfrac{1}{6}v_e^3 + \ldots .$$

This may be compared with the formula for the Doppler redshift in special relativity for a velocity of recession v_e:

$$\frac{\nu_{BA'}}{\nu_B} = (1-v_e)^{1/2}(1+v_e)^{-1/2} = 1 - v_e + \tfrac{1}{2}v_e^2 - \tfrac{1}{2}v_e^3 + \ldots .$$

For the remainder of this section we will discuss the cosmological gravitational shift given by Eq. (13). It turns out that this shift is always *blue* and becomes ever larger as the time of emission becomes earlier.

To see that this is so, we first calculate $\nu_{A'A}/\nu_{A'}$ for small distances r_e. We write R_e/R_a as a function of r_e by expanding R in a Taylor series, keeping terms to second order in the expansion velocity $H_e r_e$:

$$\begin{aligned} R_a &= R_e - \dot{R}_e\left(\frac{dt}{dr}\right)_e r_e + \tfrac{1}{2}\left[\ddot{R}_e\left(\frac{dt}{dr}\right)_e^2 + \dot{R}_e\left(\frac{d^2t}{dr^2}\right)_e\right] r_e^2 + \ldots \\ &= R_e - R_e\left(\frac{dt}{dr}\right)_e H_e r_e + \tfrac{1}{2}\left[\ddot{R}_e H_e^{-2}\left(\frac{dt}{dr}\right)_e^2 + H_e^{-1}\dot{R}_e(d^2t/dr^2)_e\right] H_e^2 r_e^2 + \ldots . \end{aligned} \tag{14}$$

Here r is the variable defined in Eq. (11), and we have used the fact that $\dot{R} = HR$. By Eq. (4), for a photon $dt/dr = -R(t)/R_e$. Then Eq. (14) may be written

$$R_a = R_e[1 + H_e r_e + \tfrac{1}{2}(\ddot{R}_e H_e^{-2} R_e^{-1} + 1) H_e^2 r_e^2 + \ldots],$$

so that

$$R_a/R_e = 1 + H_e r_e + \tfrac{1}{2}(1 - q_e) H_e^2 r_e^2 + \ldots. \tag{15}$$

Here $q_e \equiv -\ddot{R}_e R_e / \dot{R}_e^2 = -\ddot{R}_e H_e^{-2} R_e^{-1}$ is the deceleration parameter. Using Eq. (15), we find that Eq. (13) becomes

$$\frac{\nu_{A'A}}{\nu_{A'}} = 1 + \tfrac{1}{2} q_e H_e^2 r_e^2 + \ldots, \tag{16}$$

so that the cosmological gravitational shift for nearby emitters is blue ($q > 0$ in all Friedmann models with vanishing cosmological constant).

It is easy to show that, for fixed t_a,

$$\frac{d}{dt_e} \frac{\nu_{A'A}}{\nu_{A'}} = -\frac{H_e^2 R_e r_e}{R_a} (2 + q_e) \exp H_e r_e$$

for all r_e. Thus, the cosmological gravitational shift is blue for all r_e, and the earlier the time of emission, the bluer the shift.

We now show that Eq. (16) can be understood in terms of Birkhoff's theorem. According to this theorem, in a Friedmann universe the gravitational field in any sphere within which the expansion is non-relativistic is simply the Newtonian field due to the matter within the sphere. Therefore, a photon emitted by a stationary observer A' at a distance r_e from an observer A at the center of the sphere will undergo a gravitational blueshift given by

$$\frac{\nu_{A'A}}{\nu_{A'}} \approx 1 - G \int_{r_e}^{0} M(r) r^{-2} dr \tag{17}$$

in traveling to A. Here $M(r)$ is the mass interior to the radius r at the time when the photon is at the radius r.

The effect of the change of $M(r)$ with time can be neglected, because the expansion velocity is very much less than the speed of light. We then have $M(r) \simeq \frac{4}{3}\pi \varrho_e r^3$, and Eq. (17) becomes

$$\frac{\nu_{A'A}}{\nu_{A'}} \simeq 1 + \frac{GM(r_e)}{2r_e} = 1 + \tfrac{2}{3}\pi G \varrho_e r_e^2. \tag{18}$$

But in a matter-dominated Friedmann universe, $\frac{4}{3}\pi G\varrho = qH^2$ (this relation [8] can also be easily obtained from Birkhoff's theorem). Thus Eqs (18) and (16) are identical, which is what we wanted to show.

Eq. (18) yields a fractional blueshift of $\frac{2}{3}\pi G\varrho_e r_e^2$. A photon traveling from the center to the edge of a homogeneous sphere of radius r_e would undergo a redshift of identical magnitude. This same redshift was obtained by Westervelt [4] in an elegant field-theoretic calculation of the shift suffered by a photon traveling a distance r_e through a static, homogeneous medium of density ϱ_e in a Minkowski background metric. Westervelt suggested that there might be some connection between his result (which, as he emphasized, was valid only for a Minkowski background) and the cosmological shift that is actually observed. Our present paper indicates the connection: for nearby galaxies, the gravitational contribution to the cosmological shift has precisely the magnitude of Westervelt's result, although justification of this result requires Birkhoff's theorem and the assumption that the emitter is at the center of a homogeneous, spherical distribution of matter. The sign of Westervelt's result is the opposite of ours because his result gives the gravitational shift relative to the emitter, which is not observable. To find the gravitational shift seen by the absorber, Birkhoff's theorem requires us to place the absorber, rather than the emitter, at the center of the homogeneous spherical distribution of matter, so that we find a blueshift rather than a redshift.

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