ON QUANTUM ELECTRODYNAMICS IN AN EXTERNAL GRAVITATIONAL FIELD. I. CONSTRUCTION OF THE *S* MATRIX

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In this paper the influence of a prescribed unquantized gravitational field of suitable structure on the system of interacting Maxwell-Dirac fields is investigated on the basis of the *S* matrix theory.

1. Introduction

Numerous references show that quantum gravitational effects become significant not only near the Planck length but even at $10^{-13}...10^{-14}$ cm so that the cooperative effects of gravitational and strong (respectively electromagnetic) interactions should not be neglected [1, 9]. Especially the following question leads to the study of quantum electrodynamics in an external gravitational field: The number of photons of the 3K background radiation surpasses that of heavy particles in the universe by the factor 10^8. Since the creation of photons by gravitational fields occurs at an anisotropic expansion only, this mechanism of photon production does not seem to be effective enough to explain this factor. But one can imagine that particle-antiparticle pairs created by the gravitational field annihilate into photons which are changed into blackbody radiation by scattering processes. Without elaborating this problem in detail here we investigate the influence of a prescribed gravitational field of definite structure on a quantized Klein-Gordon field as well as on isolated and mutually interacting Maxwell-Dirac fields.

Some properties of the fundamental equations of quantum electrodynamics in an external gravitational field have been discussed in [8]. This system of equations is the basis for the treatment of the interaction of the Maxwell and Dirac fields by the methods of perturbation theory and the simultaneous strong consideration of the influence of the

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gravitational field, as shown in [6, 7]. In what follows we do not intend to consider the influence of the gravitational field strongly but our goal is to comprehend the electromagnetic as well as the gravitational interaction on the same perturbational basis.

2. *Suppositions for the S matrix theory*

In the conventional quantum electrodynamics coupled quantized Maxwell-Dirae fields are the subject of the dynamical evolution. Moreover the Minkowski spacetime plays the role of a background field to which the "in" and "out" states in the remote past and future are referred. The interaction is described by the evaluation of transition amplitudes with the aid of the S matrix that projects the states of an interaction-free ensemble of particles at the time $t \to -\infty$ to one at the time $t \to +\infty$. We want to attain that even under consideration of the gravitational field the fields are free of *any* interaction for large spacelike and timelike distances, this implies asymptotically a Poincarè invariant ground state and the interaction of one quantum field with the other and with the gravitational field can be described as a scattering process of plane waves. The consequence is a far reaching restriction of admissible metrics, namely:

$$\lim_{t \to \pm\infty} g_{\mu\nu}(x^a, t) = \eta_{\mu\nu} \quad \text{at every point } x^a, \tag{2.1a}$$

$$\lim_{x^a \to \pm\infty} g_{\mu\nu}(x^a, t) = \eta_{\mu\nu} \quad \text{at every time } t. \tag{2.1b}$$

Especially by this restriction we attained that the "in" and "out" operators represent creation and annihilation operators for physical particles which are associated to free fields. Therefore we do not need the concept of an "occupation number operator in an external field" (concerning the problems referring to this see e.g. [4]).

For isolated quantum fields (with respect to each other) the very restricting conditions (2.1) have been weakened [12, 13] but the interaction of these quantum fields with the external gravitational field can nevertheless be treated on the basis of perturbation theory.

As examples of gravitational fields obeying the properties (2.1) we quote:

i) *Weak gravitational fields*:

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}\,^{1}, \tag{2.2a}$$

$$g^{\mu\nu} = \eta^{\mu\nu} - \kappa h^{\mu\nu} \tag{2.2b}$$

satisfying the conditions

$$|g_{\mu\nu} - \eta_{\mu\nu}| < 1, \tag{2.3}$$

$$\lim_{t \to \pm\infty} h_{\mu\nu}(x^a, t) = 0 \quad \text{at every point } x^a, \tag{2.4a}$$

and

$$\lim_{x^a \to \pm\infty} h_{\mu\nu}(x^a, t) = 0 \quad \text{at every time } t. \tag{2.4b}$$

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[1] $\kappa \equiv \kappa_0 \hbar c$, κ_0 Einsteinian gravitational constant.

Instead of the field $h_{\mu\nu}$ we will use often the field

$$f_{\mu\nu} = h_{\mu\nu} - \tfrac{1}{2} h\eta_{\mu\nu} \quad (h \equiv \eta^{\mu\nu}h_{\mu\nu}). \tag{2.5}$$

A special case of (2.2) is the linearised Schwarzschild metric

$$f^{mn} = 0, \quad f^{44} = -f = \frac{c}{2\pi h}\frac{M}{r}. \tag{2.6}$$

In this case the condition (2.3) corresponds to

$$r > \frac{\kappa_0 M c^2}{4\pi}. \tag{2.7}$$

For further calculations we notice the Fourier transform of f:

$$f(q) = -\frac{c}{\pi h} M \frac{\delta(\omega_q)}{\boldsymbol{q}^2}. \tag{2.8}$$

Here the delta function expresses the fact that the Schwarzschild metric is static.

ii) Special Robertson-Walker metrics: Weak gravitational fields are not the only ones that permit a perturbational treatment in the way outlined above. Robertson-Walker metrics with flat space sections,

$$ds^2 = \Omega^2(t)(dx^2+dy^2+dz^2)-c^2dt^2, \tag{2.9}$$

are also suitable for that purpose if one takes care that $\Omega(t)$ becomes asymptotically static,

$$\lim_{t\to\pm\infty} \Omega(t) = 1. \tag{2.10}$$

Robertson-Walker models with this property are of methodological interest only and do not allow to make any statement concerning cosmological models with an initial singularity, of course. In the following we will perform our investigations for arbitrary functions $\Omega(t)$ satisfying the condition (2.10) but often we will specialize the results to a piecewise static course of expansion:

$$\Omega(t) = \begin{cases} 1 & \text{for } -\infty < t < -\tau, \\ \text{const} > 1 & \text{for } -(\tau-\varepsilon) < t < (\tau-\varepsilon), \\ 1 & \text{for } \tau < t < \infty. \end{cases} \tag{2.11}$$

(In the intervals $-\tau < t < -(\tau-\varepsilon)$ and $(\tau-\varepsilon) < t < \tau$ the expansion should be adiabatic and $\varepsilon \ll 2\tau$ holds.)

3. *Preliminary study: the Klein-Gordon field*

We begin with a perturbational treatment of the influence of an external gravitational field of the type (2.2) on a quantized real Klein-Gordon field. Our first goal is to determine the Lagrangian for the coupling of the Klein-Gordon field to the gravitational field. For this purpose we expand Lagrangians of the type

$$\mathscr{L} = \mathscr{L}(U_{\Omega}, U_{\Omega,\sigma}, g_{\mu\nu}, g_{\mu\nu,\varrho}) \tag{3.1}$$

in powers of κ using the metric (2.2) for $g_{\mu\nu}$ and obtain

$$\mathscr{L}(g) = \mathscr{L}(\eta) + \left.\frac{\delta \mathscr{L}}{\delta g_{\mu\nu}}\right|_{\kappa=0} \kappa h_{\mu\nu} = \mathscr{L}(\eta) - \frac{\kappa}{2} T^{\mu\nu}(\eta) h_{\mu\nu}. \tag{3.2}$$

From this we read off

$$\mathscr{L}_{\text{int}} = -\frac{\kappa}{2} T^{\mu\nu}(\eta) h_{\mu\nu}. \tag{3.3}$$

Especially for the Klein-Gordon field we get

$$\mathscr{L}_{\text{int}}^{(\text{KG})} = \frac{\kappa}{2} \frac{\hbar^2}{m_0} \left(f^{\mu\nu} \phi_{,\mu} \phi_{,\nu} + \tfrac{1}{2} \frac{m_0^2 c^2}{\hbar^2} f \phi^2 \right), \tag{3.4}$$

and the corresponding inhomogeneous Klein-Gordon equation reads

$$\eta^{\mu\nu} \phi_{,\mu,\nu} - \frac{m_0^2 c^2}{\hbar^2} \phi = \kappa h^{\mu\nu} \phi_{,\mu,\nu}. \tag{3.5}$$

As explained in Chapter 2 all the effects depending on the curvature of spacetime are introduced as perturbations on the background of the flat spacetime. Accordingly the Klein-Gordon operator formed with the Minkowski background metric occurs on the left hand side of (3.5) and the source term on the right is proportional to the coupling constant κ [3].

Following the method by Yang and Feldman [14, 5] we determine the S matrix corresponding to this inhomogeneous Klein-Gordon equation in the Heisenberg picture. This perturbational treatment with respect to the gravitational interaction is permitted under the suppositions made here because the latter is much weaker than the electromagnetic one. Following this method we obtain for the S matrix up to the first order in κ ($S = 1 + \kappa S^{(1)}$)

$$S = 1 + \frac{i\hbar\kappa}{2m_0 c} \int d^{(4)}\bar{x} \left[f^{\mu\nu} \phi_{,\mu}^{\text{in}} \phi_{,\nu}^{\text{in}} + \tfrac{1}{2} \frac{m_0^2 c^2}{\hbar^2} f (\phi^{\text{in}})^2 \right] = 1 + \frac{i}{\hbar c} \int d^{(4)}\bar{x} \mathscr{L}_{\text{int}}. \tag{3.6}$$

This S matrix describes the influence of a weak external gravitational field (2.2) on the Klein-Gordon field in terms of scattering of plane waves. If we drop the restriction to weak gravitational fields and consider e.g. the line element (2.9) in its conformally invariant form,

$$d\tilde{s}^2 = \Omega^2(t)\,(dx^2 + dy^2 + dz^2 - c^2 dt^2), \tag{3.7}$$

it is possible by means of a conformal transformation [15] to take the Klein-Gordon equation to the form

$$\frac{1}{\Omega^2} \eta^{mn} \tilde{\phi}_{,m,n} - \frac{1}{\Omega^2} \tilde{\phi}'' - \frac{m_0^2 c^2}{\hbar^2} \tilde{\phi} = 0$$

with $\tilde{\phi} \equiv \Omega\phi$, $dt = \Omega d\tilde{t}$ and $\tilde{\phi}' \equiv \partial\tilde{\phi}/\partial\tilde{t}$. With regard to (3.5) we write down this equation in the form

$$\eta^{mn}\tilde{\phi}_{,m,n} - \frac{1}{c^2}\tilde{\phi}'' - \frac{m_0^2c^2}{\hbar^2}\tilde{\phi} = \Lambda_2\left[\eta^{mn}\tilde{\phi}_{,m,n} - \frac{1}{c^2}\tilde{\phi}''\right] \tag{3.8a}$$

with

$$\Lambda_2 \equiv 1 - \frac{1}{\Omega^2(t)}, \qquad 0 \leqslant \Lambda_2 < 1. \tag{3.8b}$$

In this case the perturbation expansion is accomplished in terms of the (generally time dependent) coupling parameter Λ_2 and yields up to the first order in Λ_2

$$S = 1 - \frac{i}{2}\frac{m_0c}{\hbar}\int d^{(4)}\bar{x}\Lambda_2(t)\,[\phi^{\text{in}}(\bar{x})]^2. \tag{3.9}$$

Already here we refer to the proportionality of the Λ_2 term to the particle mass. Between the S matrix terms (3.6) and (3.9) there exists a *formal* correspondence in so far as it is possible to obtain (3.9) from (3.6) by the substitution

$$f_{\mu\nu} = -\frac{\Lambda_2}{\kappa}\eta_{\mu\nu}, \tag{3.10a}$$

$$f_{\mu\nu}(q) = -(2\pi)^{3/2}\frac{\Lambda_2(\omega_q)}{\kappa}\delta(\boldsymbol{q})\eta_{\mu\nu}. \tag{3.10b}$$

The delta function in (3.10b) reflects the flatness of the space sections of spacetime (compare (2.8)).

4. *The Maxwell-Dirac system*

i) The Mexwell field. The Lagrangian of the Maxwell field has a simpler structure than (3.1) on what account we obtain directly the following term corresponding to (3.3):

$$\mathscr{L}^{(\text{M})}_{\text{int}} = \frac{\kappa}{2}\eta^{\mu\varrho}(f^{\nu\kappa} - \tfrac{1}{4}f\eta^{\nu\kappa})B_{\varrho\kappa}B_{\mu\nu}. \tag{4.1}$$

This Lagrangian describes the influence of a gravitational field of the type (2.2) on the Maxwell field.

ii) The Dirac field. The Dirac field requires a special investigation because the Lagrangian

$$\mathscr{L}^{(\text{D})} = -\frac{\hbar c}{2}\sqrt{g}\left[\bar{\psi}\gamma^\mu\psi_{;\mu} - \bar{\psi}_{;\mu}\gamma^\mu\psi + 2\frac{m_0c}{\hbar}\bar{\psi}\psi\right] \tag{4.2}$$

does not have the simple structure (3.1). As in [2] we express the covariant bispinorial derivatives in (4.2) by the constant Dirac matrices $\gamma_{(\mu)}$ of the Minkowski spacetime and by the tetrad fields $\lambda_\mu{}^{(\nu)}$ occurring in the equations

$$\gamma_\mu = \lambda_\mu{}^{(\nu)}\gamma_{(\nu)}, \qquad g_{\mu\varrho} = \lambda_\mu{}^{(\nu)}\lambda_\varrho{}^{(\kappa)}\eta_{(\nu)(\kappa)}. \tag{4.3}$$

Then we expand the tetrad fields in powers of κ,

$$\lambda^\mu{}_{(\alpha)} = \eta^\mu{}_{(\alpha)} + \kappa k^\mu{}_{(\alpha)} \tag{4.4}$$

and comparing (4.3) with (2.2) we obtain

$$h^{\mu\varrho} = -2k^{(\mu\varrho)}$$

having used the notations [2]

$$k^{\mu\varrho} = \eta^{(\kappa)\mu}k_{(\kappa)}{}^\varrho, \qquad k^{\varrho\mu} = \eta^{(\kappa)\varrho}k_{(\kappa)}{}^\mu.$$

For the Dirac matrices the expansion (4.4) corresponds to

$$\gamma^\mu = \underset{0}{\gamma}{}^\mu - \kappa\underset{1}{\gamma}{}^\mu, \qquad \gamma_\mu = \underset{0}{\gamma}{}_\mu + \kappa\underset{1}{\gamma}{}_\mu \tag{4.5}$$

with

$$\underset{0}{\gamma}{}_\mu = \eta_\mu{}^{(\nu)}\gamma_{(\nu)}, \qquad \underset{1}{\gamma}{}_\mu = -k_{\mu\varrho}\underset{0}{\gamma}{}^\varrho, \qquad \underset{1}{\gamma}{}^\mu = k^{\varrho\mu}\underset{0}{\gamma}{}_\varrho.$$

With this we obtain from (4.2) ($m \equiv m_0c/\hbar$)

$$\mathscr{L}^{(D)}_{\text{int}} = -\frac{\kappa\hbar c}{2}\left[(k^{\nu\mu} - \tfrac{1}{2}f\eta^{\nu\mu})(\bar\psi\underset{0}{\gamma}{}_\nu\psi_{,\mu} - \bar\psi_{,\mu}\underset{0}{\gamma}{}_\nu\psi) + \bar\psi(-mf + k^{\varrho\nu}{}_{,\mu}\underset{0}{\gamma}{}^{[\mu}\underset{0}{\gamma}{}_\nu\underset{0}{\gamma}{}_{\varrho]})\psi\right].$$

It can be attained by tetrad rotations that $k^{\mu\nu}$ becomes symmetric and the Lagrangian describing the influence of an external gravitational field (2.2) on the Dirac field reduces to

$$\mathscr{L}^{(D)}_{\text{int}} = \frac{\kappa\hbar c}{4}\left[f^{\mu\nu}(\bar\psi\underset{0}{\gamma}{}_\nu\psi_{,\mu} - \bar\psi_{,\mu}\underset{0}{\gamma}{}_\nu\psi) + \frac{m_0c}{\hbar}f\bar\psi\psi\right] = -\frac{\kappa}{2}T^{\mu\nu}(\eta)h_{\mu\nu}. \tag{4.6}$$

Thus it is shown that we can get for all the three fields treated in this paper the simple expression (3.3) for $\mathscr{L}_{\text{int}}$.

iii) The interaction term of quantum electrodynamics. The interaction between the Maxwell and Dirac fields is described by

$$\mathscr{L}^{(\text{QED})} = ie_0\sqrt{g}\,g^{\mu\nu}\bar\psi\gamma_\nu A_\mu\psi.$$

From this we get with the aid of the expansions (2.2) and (4.5)

$$\mathscr{L}^{(\text{QED})} = \mathscr{L}^{(\text{QED})}_1 + \mathscr{L}^{(\text{QED})}_2 \equiv ie_0\bar\psi\underset{0}{\gamma}{}^\mu A_\mu\psi - \frac{ie_0\kappa}{2}\bar\psi(\tfrac{1}{2}f\underset{0}{\gamma}{}^\mu + f^{\mu\nu}\underset{0}{\gamma}{}_\nu)A_\mu\psi. \tag{4.7}$$

Thus the system of interacting Maxwell-Dirac fields which in their turn are influenced by an external gravitational field of the type (2.2) is described by the Lagrangian (compare [11])

$$\mathscr{L} = [\mathscr{L}^{(M)}(\eta) + \mathscr{L}^{(D)}(\eta)] + \mathscr{L}^{(\text{tot})}_{\text{int}}$$

with

$$\mathscr{L}_{\text{int}}^{(\text{tot})} = \mathscr{L}_{\text{int}}^{(\text{M})} + \mathscr{L}_{\text{int}}^{(\text{D})} + \mathscr{L}^{(\text{QED})}.$$

From this Lagrangian results the following fundamental system of equations of quantum electrodynamics in an external gravitational field of the type (2.2):

$$\underset{0}{\gamma^{\mu}}\psi_{,\mu} + \frac{m_0 c}{\hbar}\psi = \frac{ie_0}{\hbar c}\underset{0}{\gamma^{\mu}}A_{\mu}\psi + \frac{\kappa}{2}\left(f^{\mu\nu}\underset{0}{\gamma_{\nu}}\psi_{,\mu} + \frac{m_0 c}{2\hbar}f\psi\right)$$
$$- \frac{i}{2}e_0\kappa(\tfrac{1}{2}f\underset{0}{\gamma^{\mu}} + f^{\mu\nu}\underset{0}{\gamma_{\nu}})A_{\mu}\psi, \tag{4.8a}$$

and

$$A^{\varrho,\nu}{}_{,\nu} = -ie_0\bar{\psi}\underset{0}{\gamma^{\varrho}}\psi - \kappa(2^{(1)}\Gamma^{\varrho}_{\kappa\sigma}A^{\sigma,\kappa} + h^{\varrho}{}_{\sigma}{}^{,\nu}{}_{,\nu}A^{\sigma}) + \frac{ie_0\kappa}{2}\bar{\psi}(\tfrac{1}{2}f\underset{0}{\gamma^{\varrho}} + f^{\varrho\nu}\underset{0}{\gamma_{\nu}})\psi \tag{4.8b}$$

with

$$^{(1)}\Gamma^{\mu}_{\varrho\nu} \equiv \tfrac{1}{2}(h^{\mu}{}_{\varrho,\nu} + h^{\mu}{}_{\nu,\varrho} - h_{\varrho\nu}{}^{,\mu}).$$

5. *The S matrix for the Maxwell-Dirac system*

First of all we notice that from the system (4.8) result two decoupled equations for $e_0 = 0$ which describe the influence of the external gravitational field on the quantized Maxwell (respectively Dirac) field. It is possible to apply the method by Yang and Feldman to each of these two equations in the same way as to (3.5). In both cases the result is given up to the first order in κ by

$$S = 1 - \frac{i\kappa}{2\hbar c}\int d^{(4)}\bar{x}T^{\mu\nu}(\eta)h_{\mu\nu} = 1 + \frac{i}{\hbar c}\int d^{(4)}\bar{x}\mathscr{L}_{\text{int}}, \tag{5.1}$$

where we have to put in the expressions (4.1) (respectively (4.6)) for $\mathscr{L}_{\text{int}}$. We notice the analogy of (5.1) with the S matrix of the conventional quantum electrodynamics:

$$S = 1 + \frac{i}{\hbar c}\int d^{(4)}\bar{x}j^{\mu}A_{\mu} = 1 + \frac{i}{\hbar c}\int d^{(4)}\bar{x}\mathscr{L}_1^{(\text{QED})}.$$

We determine the S matrix belonging to the complete system (4.8) with the aid of a time-dependent unitary transformation $\mathfrak{U}$ in the same way as in the usual quantum electrodynamics [5]. For the solution of the equations for $\mathfrak{U}$ arising by substitution of

$$\psi = \mathfrak{U}^{-1}\psi^{\text{in}}\mathfrak{U}, \quad \mathfrak{U}_{,a} = 0, \quad \mathfrak{U}^{\dagger} = \mathfrak{U}^{-1},$$
$$A_{\mu} = \mathfrak{U}^{-1}A_{\mu}^{\text{in}}\mathfrak{U}$$

into (4.8) we make the ansatz

$$\frac{\partial\mathfrak{U}}{\partial t}\mathfrak{U}^{-1} = \frac{i}{\hbar}\int d^{(3)}x\mathscr{L}_{\text{int}}^{(\text{tot})}(x). \tag{5.2}$$

After tedious calculations one finds that this ansatz solves the equations for $\mathfrak{U}$. A difference to conventional quantum electrodynamics is that we have to pay attention to the fact that the interaction term (4.6) in $\mathscr{L}^{\text{tot}}_{\text{int}}$ contains derivatives of the fields; therefore we have to modify the equal time anticommutation relations for the Dirac field to the form

$$\left\{\bar{\psi}(x)\left(\underset{0}{\gamma^4}-\frac{\kappa}{2}f^{4\nu}\underset{0}{\gamma_\nu}\right),\ \psi(\bar{x})\right\} = -i\delta(x,\bar{x}).$$

We integrate the ansatz (5.2) with the initial condition $\mathfrak{U}(-\infty) = 1$ and obtain by iteration for $S = \mathfrak{U}(+\infty)$

$$S = 1+\frac{i}{\hbar c}\int d^{(4)}\bar{x}\mathscr{L}^{(\text{tot})}_{\text{int}}(\bar{x})-\frac{1}{(\hbar c)^2}\int d^{(4)}\bar{x}\int\limits_{-\infty}^{\bar{t}} d(c\bar{\bar{t}})\int d^{(3)}\bar{\bar{x}}\mathscr{L}^{(\text{tot})}_{\text{int}}(\bar{x})\mathscr{L}^{(\text{tot})}_{\text{int}}(\bar{\bar{x}}). \tag{5.3}$$

Since the second integrand in (5.3) is time ordered we can write it restricting ourselves to terms proportional to e_0, κ and $e_0\kappa$ in the form

$$F(\bar{x},\bar{\bar{x}}) = P\{[\mathscr{L}^{(M)}_{\text{int}}(\bar{x})+\mathscr{L}^{(D)}_{\text{int}}(\bar{x})]\mathscr{L}^{(QED)}_{1}(\bar{\bar{x}})\}+P\{\mathscr{L}^{(QED)}_{1}(\bar{x})\,[\mathscr{L}^{(M)}_{\text{int}}(\bar{\bar{x}})+\mathscr{L}^{(D)}_{\text{int}}(\bar{\bar{x}})]\}$$

with $F(\bar{x},\bar{\bar{x}}) = F(\bar{\bar{x}},\bar{x})$. Because of this property we are allowed to change the boundaries in the second integral of (5.3) and obtain [5]

$$S = 1+\frac{i}{\hbar c}\int d^{(4)}\bar{x}\mathscr{L}^{(\text{tot})}_{\text{int}}(\bar{x})-\tfrac{1}{2}\frac{1}{(\hbar c)^2}\int d^{(4)}\bar{x}\int d^{(4)}\bar{\bar{x}}\mathscr{L}^{(\text{tot})}_{\text{int}}(\bar{x})\mathscr{L}^{(\text{tot})}_{\text{int}}(\bar{\bar{x}}).$$

In this way the following double series in e_0 and κ results for the S matrix in question[2]:

$$S = 1-$$

$$-\frac{e_0}{\hbar c}\int d\bar{x}\bar{\psi}\gamma^\mu A_\mu\psi \tag{5.4a}$$

$$+\frac{i\kappa}{4}\int d\bar{x}[f^{\mu\nu}(\bar{\psi}\gamma_\nu\psi_{,\mu}-\bar{\psi}_{,\mu}\gamma_\nu\psi)+mf\bar{\psi}\psi] \tag{5.4b}$$

$$+\frac{i\kappa}{2\hbar c}\int d\bar{x}[f^{\alpha\beta}B^\mu{}_\beta B_{\mu\alpha}-\tfrac{1}{4}fB^{\mu\nu}B_{\mu\nu}] \tag{5.4c}$$

$$+\frac{e_0\kappa}{2\hbar c}\int d\bar{x}\bar{\psi}(\tfrac{1}{2}f\gamma^\mu+f^{\mu\nu}\gamma_\nu)A_\mu\psi \tag{5.4d}$$

$$-\frac{ie_0\kappa}{8\hbar c}\int d\bar{x}\int d\bar{\bar{x}}P\{[f^{\mu\nu}(\bar{\psi}\gamma_\nu\psi_{,\mu}-\bar{\psi}_{,\mu}\gamma_\nu\psi)+mf\bar{\psi}\psi]\bar{\psi}(\bar{\bar{x}})\gamma^\varrho A_\varrho(\bar{\bar{x}})\psi(\bar{\bar{x}})\} \tag{5.4e}$$

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[2] In these terms occur the fields ψ^{in}, A^{in}_μ etc. The superscript "in" as well as the index „0“ of the Dirac matrices have been omitted for formal clearness.

$$-\frac{ie_0\kappa}{8\hbar c}\int d\bar{x}\int d\bar{\bar{x}}P\{\bar{\psi}(\bar{x})\gamma^\varrho A_\varrho(\bar{x})\psi(\bar{x})\,[f^{\mu\nu}(\bar{\psi}\gamma_\nu\psi_{,\mu}-\bar{\psi}_{,\mu}\gamma_\nu\psi)+mf\bar{\psi}\psi]\}\tag{5.4f}$$

$$-\frac{ie_0\kappa}{4(\hbar c)^2}\int d\bar{x}\int d\bar{\bar{x}}P\{[f^{\alpha\beta}B^\mu{}_\beta B_{\mu\alpha}-\tfrac{1}{4}fB^{\mu\nu}B_{\mu\nu}]\bar{\psi}(\bar{\bar{x}})\gamma^\varrho A_\varrho(\bar{\bar{x}})\psi(\bar{\bar{x}})\}\tag{5.4g}$$

$$-\frac{ie_0\kappa}{4(\hbar c)^2}\int d\bar{x}\int d\bar{\bar{x}}P\{\bar{\psi}(\bar{x})\gamma^\varrho A_\varrho(\bar{x})\psi(\bar{x})\,[f^{\alpha\beta}B^\mu{}_\beta B_{\mu\alpha}-\tfrac{1}{4}fB^{\mu\nu}B_{\mu\nu}]\}.\tag{5.4h}$$

In order to obtain the system of equations of quantum electrodynamics analogous to (4.8) for a metric of the type (3.7) we start from the generally covariant wave equation with the Dirac current and the generally covariant Dirac equation with electromagnetic field. From this we get by conformal transformations the equations

$$A^{\mu,\nu}{}_{,\nu} = ie_0(1+3A_1)\bar{\psi}\underset{0}{\gamma^\mu}\psi\tag{5.5a}$$

and

$$\underset{0}{\gamma^\mu}\psi_{,\mu}+\frac{m_0c}{\hbar}\psi=\frac{ie_0}{\hbar c}\underset{0}{\gamma^\mu}A_\mu\psi-A_1\frac{m_0c}{\hbar}\psi.\tag{5.5b}$$

The parameter A_1 given by

$$A_1 = 1-\frac{1}{\Omega(t)},\qquad 0\leqslant A_1<1$$

is adapted to the Dirac equation with its first order derivatives and consequently the coupling by the factor $3A_1$ in (5.5a) is an additional approximation. The promptest way to obtain the corresponding S matrix is that of the formal substitution (3.10a) with A_1 in place of A_2 which gives

$$S = 1-$$

$$-\frac{e_0}{\hbar c}\int d\bar{x}\psi\gamma^\mu A_\mu\psi\tag{5.6a}$$

$$-im\int d\bar{x}A_1(\bar{t})\bar{\psi}\psi\tag{5.6b}$$

$$-\frac{3e_0}{\hbar c}\int d\bar{x}A_1(\bar{t})\bar{\psi}\gamma^\mu A_\mu\psi\tag{5.6c}$$

$$+\frac{ie_0m}{2\hbar c}\int d\bar{x}A_1(\bar{t})\int d\bar{\bar{x}}P\{\bar{\psi}(\bar{x})\psi(\bar{x})\bar{\psi}(\bar{\bar{x}})\gamma^\mu A_\mu(\bar{\bar{x}})\psi(\bar{\bar{x}})\}\tag{5.6d}$$

$$+\frac{ie_0m}{2\hbar c}\int d\bar{x}\int d\bar{\bar{x}}A_1(\bar{\bar{t}})P\{\bar{\psi}(\bar{x})\gamma^\mu A_\mu(\bar{x})\psi(\bar{x})\bar{\psi}(\bar{\bar{x}})\psi(\bar{\bar{x}})\}.\tag{5.6e}$$

In (5.6b, d, e) we made use of the fact that the fields ψ^{in} and $\bar{\psi}^{\text{in}}$ fulfil the homogeneous Dirac equation. The proportionality to m_0 which came about during this procedure is a consequence of the conformal invariance of the massless Dirac equation. The following properties of the S matrix terms (5.4) and (5.6) are obvious:

i) In the form of (5.4a) (respectively (5.6a)) the conventional quantum electrodynamics results up to the first order for $\kappa = 0$, ($\Lambda_1 = 0$);

ii) For $e_0 = 0$ the Maxwell and Dirac fields decouple and analogously to (3.6) the influence of the external gravitational field on each of these two fields remains in the form of (5.4b, c);

iii) Proportional to $e_0\kappa$ ($e_0\Lambda_1$) there occur both processes of first order, (5.4d) and (5.6c), and of second order, namely (5.4e...h) and (5.6d, e);

iv) For conformally flat metrics $f^{\mu\nu} \sim \eta^{\mu\nu}$ holds. Therefore the terms (5.4c, g, h) disappear, that is to say the Maxwell field which is isolated from the Dirac field does not couple to conformally flat gravitational fields. Accordingly the terms (5.4c, g, h) have no counterpart in (5.6).

6. *Evaluation of the S matrix with the aid of Wick's theorem*

The processes occurring in (5.4d...h) represent the corrections to the processes in (5.4a) arising from the gravitational field ((5.6c...e) likewise). In order to make the content of the S matrix in terms (5.4d...h) (respectively (5.6c...e)) accessible we have to decompose them with the aid of Wick's theorem [5] into a sum of normal products such that the operators on the right in each term of the sum annihilate the particles of the initial state and those on the left create the particles of the final state. In comparison with the conventional quantum electrodynamics two peculiarities occur in the present case: (*i*) Not only one graph but a whole series of graphs contribute to a definite matrix element, namely one from each of the terms (5.4d...h) (respectively (5.6c...e)); (*ii*) One has to form not only contractions between the operators but also between the operators and their derivatives. The latter result in the same way as the analogous expressions in meson electrodynamics [10]:

$$\langle 0, \text{in}|T[\psi^{\text{in}}{}_{,\mu}(x)\bar{\psi}^{\text{in}}(\bar{x})]\,|0, \text{in}\rangle = \frac{i}{2}\,\partial_\mu S^{(\text{F})}(x-\bar{x}),$$

$$\langle 0, \text{in}|T[\psi^{\text{in}}(x)\bar{\psi}^{\text{in}}{}_{,\mu}(\bar{x})]\,|0, \text{in}\rangle = \frac{i}{2}\,\partial_{\bar{\mu}} S^{(\text{F})}(x-\bar{x}),$$

and

$$\langle 0, \text{in}|T[A^{\text{in}}{}_{\mu,\nu}(x)A^{\text{in}}_{\varrho}(\bar{x})]\,|0, \text{in}\rangle = -\frac{i\hbar c}{2}\,\eta_{\mu\varrho}\partial_{\bar{\nu}} D^{(\text{F})}(x-\bar{x}).$$

7. *Summary*

In the simplest case the perturbational treatment of the influence of the gravitational field on quantized fields is restricted to metrics which permit the introduction of the gravitationally induced effects on the background of flat spacetime. For mutually isolated quantum fields the construction of the S matrix follows the method by Yang and Feldman. Restricting to the first order in the coupling constant κ for all the fields under consideration we get

$$S = 1 - \frac{i\kappa}{2\hbar c}\int d^{(4)}x T^{\mu\nu}(\eta)h_{\mu\nu}.$$

The S matrix for the interaction of quantized fields with each other and with the external gravitational field constructed with the aid of a time dependent unitary transformation is a double series in the coupling constants e_0 and κ. Proportional to $e_0\kappa$ there are first order processes as well as second order ones. Since the corresponding interaction Lagrangian contains derivatives of the fields we have to form contractions between the operators and their derivatives, too. While in conventional quantum electrodynamics to each matrix element corresponds one Feynman graph, in the present case to each process contributes a whole series of graphs. The method allows in principle to evaluate each process to any desired order.

From the S matrix formalism it results clearly that no interaction takes place between massless particles and those gravitational fields which are described by conformally flat line elements. In this case the terms (5.4c) and (5.6b) disappear for $m_0 = 0$ and the S matrix (3.9) reduces to $S = 1$.

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