INCLUSIVE COULOMB SCATTERING IN THE FOURTH ORDER OF QED

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Using the dimensional regularization the inclusive cross section for the Coulomb scattering is obtained. It is shown that it gives the same result as the Pauli-Villars regularization. At the end we present some numerical results for the inclusive Coulomb scattering cross-section.

1. *Introduction*

Calculations of radiative corrections in quantum electrodynamics are plagued by the well known infrared divergence problem. Conventionally, one overcomes this difficulty by regularizing the infrared divergences at the S-matrix level by giving the photon an infinitesimal mass λ. Such divergences arise in both virtual photonic corrections and the real Bremsstrahlung processes. Fortunately when we sum up probabilities from these two sources the infrared divergences cancel each other in the arbitrary order of perturbation theory, and the regularizing parameter λ can be set equal to zero. It was shown in [1] that it is possible to calculate infrared divergent integrals without the introduction of a fictitious photon mass. One can do it using the dimensional regularization method. One parameter allows us to regularize ultraviolet and infrared divergences. In both cases these divergences manifest themselves as poles for $n = 4$. It was shown in [1] that the elastic and the inelastic poles cancel each other. In Section 2 we calculate, using the dimensional regularization, the cross section for the Coulomb scattering and show that the result obtained from the dimensional and the Pauli-Villars regularizations are the same up to the fourth order of perturbation theory. In Section 3 we give the expression for the cross section of the inclusive Coulomb scattering which is free from divergences. At the end we present some numerical calculations.

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2. *Coulomb scattering*

It was shown [2–4] that for the zero photon mass, non-zero transition probabilities are obtained by summing the probabilities of the processes for which the final states differ solely as to the number of soft photons they contain. Such an overall probability can be written symbolically as

$$\sum_m \int P(p, q, k; k_1', k_2', \ldots, k_m')$$

in which the summation and integration run over the states of the soft photons. This formula is simplified significantly in perturbation theory since only the emission of several soft photons is possible in the lower orders of the theory. In the lowest non-trivial order of perturbation theory it is possible to have either an elastic process without the emission of soft photons or a process with the emission of a single photon. In the simplest case of electron scattering in an external field, the aforementioned overall probability is given by

$$P(p, q) + \int_{|\vec{k}| \leqslant \Delta E} d\Gamma P(p, q, k).$$

The divergent expressions of the type $1/(n-4)$ in both terms of the sum cancel each other and in the overall probability we may pass to the limit when $n \to 4$, obtaining the finite result.

Now we compute the cross section for the electron scattering in the Coulomb field. Figs 1, 2 and 3 contain diagrams which we must take into consideration [2, 5, 11]. The last four diagrams of Fig. 2 renormalize the mass and the charge of the electron. In this procedure no renormalization of field operators is needed. Moreover, these diagrams are convenient in the discussion of the infrared catastrophy because one can show which infrared divergences of the elastic processes are cancelled by those of the inelastic processes. For instance the infrared divergence of Fig. 2a is cancelled by that of Fig. 3a. Since we use the dimensional regularization, we must generalize the three-dimensional integral in Fig. 3 to $n-1$ dimensions. To obtain this one can use the renormalized vertex correction.

The theory of the dimensional regularization can be found in [6, 7]. But we have to change this definition of the regularization because in the intermediate steps of calculations we encounter terms which have no physical meaning. For instance from the vertex correction we obtain the expression

$$\ln E\left(\left(\frac{\sqrt{1+\lambda^2}}{\lambda} + \frac{\lambda}{\sqrt{1+\lambda^2}}\right) \ln(\sqrt{1+\lambda^2} + \lambda) - 1\right),$$

which has no meaning because we have the logarithm of the dimensional quantity E. From the inelastic processes we have

$$-\ln \Delta E\left(\left(\frac{\sqrt{1+\lambda^2}}{\lambda} + \frac{\lambda}{\sqrt{1+\lambda^2}}\right) \ln(\sqrt{1+\lambda^2} + \lambda) - 1\right)$$

and when we sum up we obtain the correct expression. To avoid this we will use the following dimensional regularization for the virtual corrections

$$\frac{d^4k}{(2\pi)^4} \to \mu^{4-n} \frac{d^nk}{(2\pi)^n},$$

where μ is an arbitrary parameter with the dimension of the mass. Using this definition of the regularization procedure for virtual corrections one can find the dimensional regulariza-

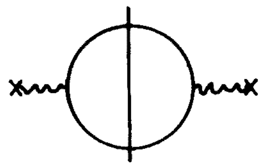

Fig. 1

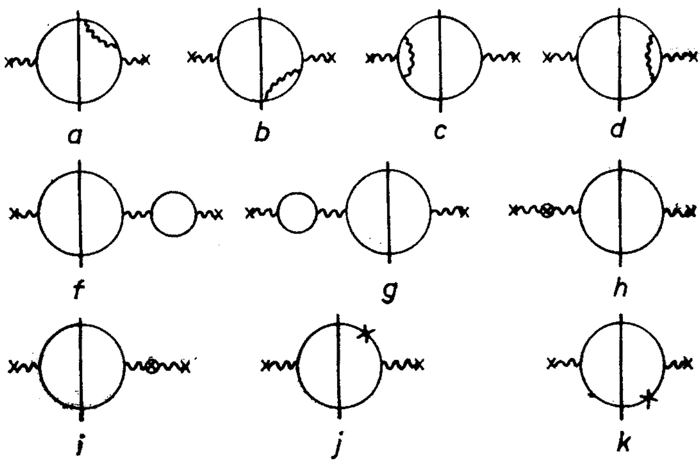

Fig. 2

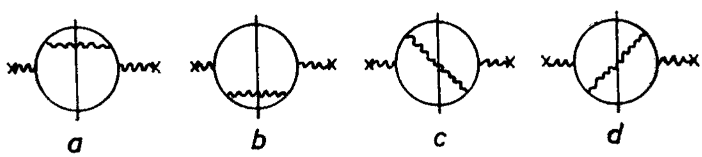

Fig. 3

ion for the real inelastic processes. The generalization of the momentum space integral s [1, 8]

$$\frac{d^3k}{(2\pi)^3 2k_0} \to \mu^{4-n} \frac{d^{n-1}K}{(2\pi)^{n-1} 2|\vec{K}|},$$

$$|\vec{K}| = K_0 = (K_1^2 + K_2^2 + \dots + K_{n-1}^2)^{1/2}. \tag{1}$$

Since μ is an arbitrary parameter, the physical quantities like cross sections cannot depend on it. The final result for the Coulomb scattering is (see Appendix)

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_0 \left\{1 - \frac{2\alpha}{\pi}\left[\ln\frac{E}{\Delta E}\left(\left(\frac{\sqrt{1+\lambda^2}}{\lambda} + \frac{\lambda}{\sqrt{1+\lambda^2}}\right)\ln(\sqrt{1+\lambda^2}+\lambda)-1\right)+D\right]\right\}. \quad (2)$$

This is the same result which was obtained by Schwinger [9, 10]. Thus we see that the dimensional and the Pauli-Villars regularizations give the same result. But the dimensional regularization is more convenient because all calculations are simpler and the Ward identities are fulfilled automatically.

3. *Inclusive cross section*

In this Section we calculate the inclusive cross section for the Coulomb scattering. The cross section with ΔE for large energy becomes negative and it is very difficult to interpret the result. The inclusive cross section is positive and for large energy we obtain a compact expression for it.

Now we give the result for the inclusive cross section. To calculate it we must sum up the same diagrams but the integration over the energy of the final photon is from zero to the maximal value $E-m$. Thus we cannot use the approximation that the energy of the photon $\omega \ll m$. The inclusive cross section is (see Appendix)

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{incl}} = \left(\frac{d\sigma}{d\Omega}\right)_0 \left\{1 - \frac{2\alpha}{\pi}\left[\ln\frac{E}{E-m}\left(\left(\frac{\sqrt{1+\lambda^2}}{\lambda} + \frac{\lambda}{\sqrt{1+\lambda^2}}\right)\right.\right.\right.$$

$$\left.\left.\left.\times\ln(\sqrt{1+\lambda^2}+\lambda)-1\right)+D\right]+\Delta\right\}. \quad (3)$$

All integrals in this expression are finite.

It is rather difficult to say something about this cross section for an arbitrary energy of the initial electron. But we can obtain a compact result if we calculate the cross section asymptotically, i. e., for $E \gg m$. For large E, Δ is a function only of the angle θ. Then asymptotically we have

$$\left(\frac{d\sigma}{d\Omega}\right)^{\text{as}}_{\text{incl}} = \left(\frac{d\sigma}{d\Omega}\right)^{\text{as}}_0 \left\{1 + \frac{2\alpha}{\pi}\left[F(\theta)+\tfrac{13}{6}\,s\right]\right\}, \quad (4)$$

where $s = \ln E/m$. From this calculation it follows that in the fourth order of perturbation theory we have for the asymptotic region

$$\left(\frac{d\sigma}{d\Omega}\right)^{\text{as}}_{\text{incl}} = \left(\frac{d\sigma}{d\Omega}\right)^{\text{as}}_{\text{el}} + \left(\frac{d\sigma}{d\Omega}\right)^{\text{as}}_{\text{nel}},$$

where $(d\sigma/d\Omega)^{as}_{el}$ is the contribution from the elastic processes and $(d\sigma/d\Omega)^{as}_{nel}$ is the one from the inelastic processes (we cancelled the infrared poles).

$$\left(\frac{d\sigma}{d\Omega}\right)^{as}_{el} = \left(\frac{d\sigma}{d\Omega}\right)^{as}_{0}\left\{1+\frac{2\alpha}{\pi}\left[F_{el}(\theta)+\tfrac{2}{3}s+H(\theta,s)\ln\frac{E}{\mu}\right]\right\},$$

$$\left(\frac{d\sigma}{d\Omega}\right)^{as}_{nel} = \frac{2\alpha}{\pi}\left\{\left(\frac{d\sigma}{d\Omega}\right)^{as}_{0}\left[F_{nel}(\theta)+\tfrac{3}{2}s-H(\theta,s)\ln\frac{E}{\mu}\right]\right\},$$

and when we add those expressions we obtain (4). From (3) it is clear that one can obtain the inclusive cross section from (2), changing the resolution energy $\Delta E \to E-m$, and adding a function Δ. But asymptotically Δ depends only on θ. We will have a similar situation in higher orders of perturbation theory. In the $2n$-th order we must add the expression

$$\left(\frac{d\sigma}{d\Omega}\right)_0 \alpha^{n-1}\Delta^n(\theta, E, m).$$

We conjecture that asymptotically

$$\Delta^n(\theta, E, m) = \sum_{i=0}^{n-2} f_i(\theta)s^i$$

and therefore the leading term in the asymptotic expansion in the $2n$-th order can be determined from elastic processes and soft inelastic processes if one formally changes $\Delta E \to E$.

4. *Numerical calculations*

In this Section we present some numerical results for the inclusive Coulomb scattering cross section. Since this cross section for a very small momentum is equal to $(d\sigma/d\Omega)_0$, one sees that the virtual and real corrections will be significant for large energies. We define the quantity

$$\sum(\theta, s) = \left(\frac{d\sigma}{d\Omega}\right)_{incl}\Big/\left(\frac{d\sigma}{d\Omega}\right)_0.$$

For a fixed s, $\Sigma(\theta, s)$ is a growing function of θ and also for a fixed θ it is a growing function of s. Figs 4 and 5 present plots of the $\Sigma(\theta, s)$ for given θ and s, respectively. The function $F(\theta)$ approaches infinity for $\theta = 180°$. However, the inclusive cross section for this angle is finite because from $(d\sigma/d\Omega)^{as}_0$ we have the factor $\cos^2\theta/2$ and the product $F(\theta)\cdot\cos^2\theta/2$ is finite for $\theta = 180°$.

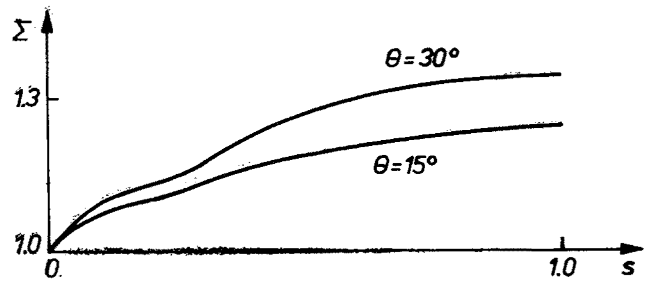

Fig. 4

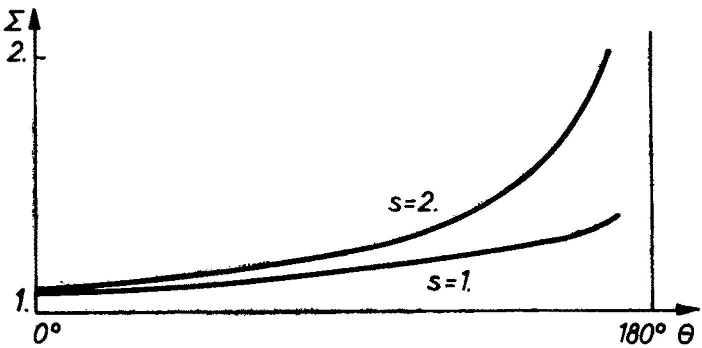

Fig. 5

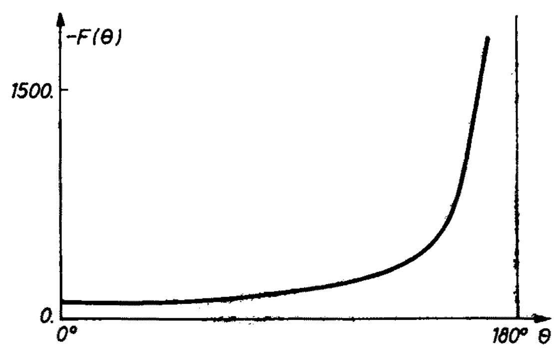

Fig. 6

5. *Conclusion*

We calculated in this paper the Coulomb scattering using the dimensional regularization, showing that the result is the same as that obtained from the Pauli-Villars regularization. We showed also that one can obtain the inclusive Coulomb scattering from the expression with a finite resolution energy, changing ΔE into $E-m$ and adding a function

Δ which does not depend on s asymptotically. We think that this result can be generalized to higher orders of perturbation theory, i. e., in the $2n$-th order Δ^n does not depend on s^{n-1}.

The author would like to thank Professor I. Białynicki-Birula for suggesting this problem and critical discussions.

APPENDIX

Calculating the cross section (2) we use the following relation

$$\frac{1-\dfrac{1}{2\beta v}\ln\dfrac{1+\beta v}{1-\beta v}}{1+\lambda^2(1-x^2)} = \tfrac{1}{2}\cdot\frac{\ln\left(1+\lambda^2(1-x^2)\right)}{1+\lambda^2(1-x^2)} - \frac{\ln\dfrac{2E}{m}-1}{1+\lambda^2(1-x^2)}$$

$$+\frac{m^2}{2\beta vE^2}\left[\frac{\ln\dfrac{1-\beta v}{2}}{1+\beta v}-\frac{\ln\dfrac{1+\beta v}{2}}{1-\beta v}\right],$$

where

$$\lambda = \frac{p}{m}\sin\frac{\theta}{2}, \qquad v^2 = 1-(1-x^2)\sin^2\frac{\theta}{2}.$$

We also used

$$\psi(x) = \frac{d}{dx}\ln\Gamma(x), \qquad \psi(1)-\psi(\tfrac{1}{2}) = 2\ln 2.$$

$\Gamma(x)$ is Euler's function and we utilized the following integral representation of the hypergeometrical function

$$F(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt,$$

$$\operatorname{Re} c > \operatorname{Re} b > 0, \qquad |\arg(1-z)| < \pi.$$

Also in (2) we have

$$\left(\frac{d\sigma}{d\Omega}\right)_0 = \frac{Q^2\alpha}{16\pi p^2\beta^2\sin^4\dfrac{\theta}{2}}\left(1-\beta^2\sin^2\frac{\theta}{2}\right),$$

$$D = -\frac{m^2}{\beta E^2}(\lambda^2+\tfrac{1}{2})\int_0^1\frac{dx}{v}\left[\frac{\ln\dfrac{1-\beta v}{2}}{1+\beta v}-\frac{\ln\dfrac{1+\beta v}{2}}{1-\beta v}\right]-\frac{1}{3\lambda^2}+\tfrac{14}{9}$$

$$+\frac{m^2}{2\beta E^2}\left[\frac{\ln\frac{1-\beta}{2}}{1+\beta}-\frac{\ln\frac{1+\beta}{2}}{1-\beta}\right]+\left(\frac{m^2}{2E^2\left(1-\beta^2\sin^2\frac{\theta}{2}\right)}-\frac{5}{2}\right.$$

$$\left.+\frac{(1+\lambda^2)^2}{3\lambda^4}-\frac{2}{\lambda^2}\right)\frac{\lambda}{\sqrt{1+\lambda^2}}\ln(\sqrt{1+\lambda^2}+\lambda).$$

Calculating (3) we encountered the integrals

$$A_{m,n}=\int d\Omega_k(1-\vec{k}\vec{v})^{-m}(1-\vec{k}\vec{u})^{-n}.$$

These integrals can be calculated for $m \geqslant 0$ and $n \geqslant 0$, applying the Feynman-Schwinger technique (m, n are integers). But we met also the integral $A_{2,-1}$ which can be expressed by the above using

$$A_{2,-1}=\{(\vec{v}\vec{u})/v^2\}A_{1,0}+\{1-[(\vec{v}\vec{u})/v^2]\}A_{2,0}.$$

In (3) we have

$$\left(\frac{d\sigma}{d\Omega}\right)_0\cdot\Delta=\frac{Q^2\alpha^2m^2E}{2\pi^2}\int_0^1 dxdy\int_0^{E-m}\frac{d\omega}{\omega}\left\{\left[\frac{2byp'(E^2+E'^2)}{m^2\omega(b^2-\varrho^2)^2}-\frac{4pyE^2}{m^2\omega b_0^3}\right]\right.$$

$$+\frac{1}{2m^2}\left[\frac{p}{E^2-\eta_0^2}-\frac{p'}{a^2-\eta^2}\right]+\frac{1-x}{\omega}\left[\frac{p'c}{(c^2-\gamma^2)^2}-\frac{p}{c_0^3}\right]$$

$$+\frac{1-x}{\omega}\left[\frac{p'd}{(d^2-\delta^2)^2}-\frac{p}{c_0^3}\right]-\frac{2y}{\omega}\left[\frac{p'b}{(b^2-\varrho^2)^2}-\frac{p}{b_0^3}\right]$$

$$+\frac{E^2x(1-x)}{\omega^2}\left[\frac{3p}{c_0^4}-\frac{p'(3d^2-\delta^2)}{(d^2-\delta^2)^3}\right]$$

$$+\frac{x(1-x)}{\omega^2}\left[\frac{3E^2p}{c_0^4}-\frac{p'E'^2(3c^2-\gamma^2)}{(c^2-\gamma^2)^3}\right]$$

$$+\frac{2Ey(1-y)}{\omega^2}\left[\frac{p'E'(3b^2-\varrho^2)}{(b^2-\varrho^2)^3}-\frac{3Ep}{b_0^4}\right]-\frac{p}{m^2(c^2-\gamma^2)}$$

$$+\frac{p'}{m^2(d^2-\delta^2)}-\frac{2xp'\vec{p}'\vec{\gamma}}{m^2\gamma^2(c^2-\gamma^2)}+\frac{2xp'c^2(E'\gamma^2-c\vec{p}'\vec{\gamma})}{m^2\gamma^2(c^2-\gamma^2)^2}$$

$$\left.+\frac{2xp'\vec{p}\vec{\delta}}{m^2\delta^2(d^2-\delta^2)}-\frac{2xp'd^2(E\delta^2-d\vec{p}\vec{\delta})}{m^2\delta^2(d^2-\delta^2)^2}\right\},$$

where

$$E' = E-\omega, \quad p'^2 = p^2+\omega^2-2E\omega,$$

$$a = xE+(1-x)E', \quad \vec{\eta} = x\vec{p}+(1-x)\vec{p}',$$

$$b = yxE+y(1-x)E'+\frac{1-y}{\omega}(p^2+p'^2+\omega^2-2\vec{p}\vec{p}'),$$

$$\vec{\varrho} = -(yx+2-2y)\vec{p}-(3y-yx-2)\vec{p}',$$

$$c = (1-x)E+\frac{x}{\omega}(p^2+p'^2+\omega^2-2\vec{p}\vec{p}'),$$

$$\vec{\gamma} = -(1+x)\vec{p}+2x\vec{p}',$$

$$d = (1-x)E'+\frac{x}{\omega}(p^2+p'^2+\omega^2-2\vec{p}\vec{p}'),$$

$$\vec{\delta} = -2x\vec{p}+(3x-1)\vec{p}',$$

$$\eta_0^2 = p^2(x^2+(1-x)^2+2x(1-x)\cos\theta),$$

$$b_0^2 = \frac{2(1-y)p^2(1-\cos\theta)}{\omega}, \quad c_0^2 = \frac{2xp^2(1-\cos\theta)}{\omega}.$$

REFERENCES

[1] R. Gastmans, R. Meuldermans, *Nucl. Phys.* **B63**, 277 (1973).
[2] J. M. Jauch, F. Rohrlich, *The Theory of Photons and Electrons*, Addison-Wesley Publishing Company, 1959.
[3] D. R. Yennie, S. Frautschi, H. Suura, *Ann. Phys.* (N.Y.) **13**, 379 (1961).
[4] K. E. Eriksson, *Nuovo Cimento* **19**, 1010 (1961).
[5] I. Białynicki-Birula, *Phys. Rev.* **D2**, 2877 (1970).
[6] G. 'tHooft, M. Veltman, *Nucl. Phys.* **B44**, 189 (1972).
[7] G. Leibbrandt, *Rev. Mod. Phys.* **47**, 849 (1975).
[8] W. J. Marciano, A. Sirlin, *Nucl. Phys.* **B88**, 86 (1975).
[9] J. Schwinger, *Phys. Rev.* **75**, 898 (1949).
[10] J. Schwinger, *Phys. Rev.* **76**, 790 (1949).
[11] I. Białynicki-Birula, Z. Białynicka-Birula, *Quantum Electrodynamics*, Pergamon Press, Oxford 1974.