MULTIPLE SCATTERING FROM NON-OVERLAPPING POTENTIALS

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Scattering from a collection of N fixed non-overlapping scatterers has been considered in a framework which uses as input the corresponding two-body on-shell scattering amplitudes. By introducing a differential operators technique we have been able to reduce the underlying multiple scattering equations to a system of algebraic equations. We exemplify the analysis by discussing a variety of situations for which practical solutions have been known in the literature and show that our method reproduces and generalizes the results obtained by other authors. In particular, our equations are completely equivalent to those obtained in partial wave basis. We turn next to the most interesting high energy scattering case where the partial wave method is impractical and propose two non-eikonal approximation schemes: (*i*) the never-come-back approximation which neglects reflections and is designed for small angles scattering; and, (*ii*) the large separation approximation where the relevant expansion parameter is taken to be the ratio of the projectile wavelength to the mutual separation between the scatterers. The latter framework may be regarded as a complementary approach to the former because it includes reflections to all orders and its validity is not restricted to small angles scattering. Furthermore, if the two body amplitude is given in terms of l_{max} partial waves, the large separation method becomes exact after $(2\,l_{max}+1)$ iterations.

1. Introduction

The concept of replacing the nuclear target by a collection of fixed scatterers (fixed scatterer approximation, hereafter referred to as FSA) has a long history of application in the study of scattering of hadrons from nuclei. Undoubtedly, the simplest and the most prominent example of such application is the research based on the Glauber theory [1] which has offered remarkably accurate fits at asymptotic energies and fixed momentum transfer. Apart from the question of the agreement with experiment, the advantageous feature of the FSA is the comparative ease of computations which permits model problems to be solved more completely than it would be possible for a more sophisticated theory possessing the full complexity inherent to the many body problem.

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In the early papers published prior to the ultimate clarification of the connection between the FSA and the exact theory, the FSA approach had been employed on a somewhat intuitive basis. In their illuminating paper Foldy and Walecka [2] have explicitly shown that FSA emerges from exact theory upon performing closure over all intermediate nuclear excitations (assumed degenerate). Furthermore, if the mean excitation energy can be regarded to be negligible as compared with the incident energy, the genuine many body scattering problem reduces to that of a scattering from an assembly of fixed potentials. The projectile-target scattering amplitude is then identified, by an extra assumption of the theory, with the FSA amplitude averaged with respect to the nuclear ground state wave function.

Unfortunately, the FSA alone is, in general, not sufficient to make the scattering problem fully tractable. The difficulties are twofold. Firstly, what one usually knows are not the projectile-nucleon potentials but rather the corresponding scattering amplitudes. Even with perfect measurements (which they are not) of the cross section, polarizations, etc. the two body scattering data can provide us only with the on-shell amlitudes whereas in the scattering from many centres, in general, one needs also the off-shell amplitudes and this missing bit of information has to be pluged in. By and large the off-shell continuation is a major problem in the scattering from complex systems. Although the above ambiguity would have disappeared if we had had the microscopic potentials, but even in that case there would persist a second difficulty connected with the fact that the total nuclear potential is a highly non-central operator. Consequently, the different angular momenta are mixed and one has to cope with an extremely complicated multichannel situation. Hence, even the simplest model considerations such as scattering from a set of potentials lead to considerable computational difficulties (we are not aware of any realistic calculation which would go beyond a two body target [3]).

Of particular interest are theories assuming that the underlying microscopic potentials are of separable form which may be regarded as a very convenient but otherwise rather arbitrary prescription for the off-shell continuation. Since a separable potential is operative in only one partial wave there is no mixing problem in this model and the resulting scheme is solvable. A formal solution of the multiple scattering equations when the microscopic interactions are of separable form has been given by Foldy and Walecka [2].

The off-shell difficulty can be also evaded under specific circumstances when the scatterers are not only fixed but the underlying potentials do not overlap. In this case the projectile travelling from one scatterer to another moves in a free space where the appropriate wave function is completely determined by the on-shell scattering amplitude. Thus, at the expense of the additional non-overlap assumption the problem is solvable in terms of on-shell amplitudes.

The first quantum mechanical formulation of multiple scattering seems to be due to Foldy [4] who actually derived the multiple scattering equations for the particular case of s-wave scattering. His work has been generalized by Lax [5] but both these authors introduced their equations merely on the grounds of physical plausibility and in fact failed to produce a practical solution to the problem. The explicit solution for a two body target with either s-wave, or p-wave interaction has been given by Brueckner [6]. Contrary to

some assertions in the literature his solution is not restricted to zero-range forces and is exact within the framework of non-overlapping interactions generated by infinitely heavy scatterers. On the theoretical side, the first rigorous derivation of the multiple scattering equation was given by Watson [7] who also showed that they indeed represent formal solution to the many body problem. Brueckner's result for the scattering off a deuteron has been extended to comprise all partial waves but the solution is no longer exact; it has been obtained [8] for high energy scattering where the wavelength of the incident particle is small in comparison with the separation of the two centres. Another solution for a two body target, also valid for any number of partial waves, has been given by Beg [9] but his solution is restricted to single and double scattering terms. A complete solution of the multiple scattering equations for non-overlapping potentials has been obtained by Agassi and Gal [10] in the partial wave basis. Although the latter formulation is exact, its applications for economy reasons have been restricted to situations where only a few partial waves contribute.

Explicit practical solutions to the multiple scattering equations have been obtained only for the lightest nuclei. Larger nuclei pose considerable computational difficulties and the number of equations rapidly becomes unmanagable. The situation is additionally aggravated by the difficulties to carry through the multidimensional integrations arising from averaging of the FSA amplitude over the nuclear density. It is essentially for the latter complicacy that many authors eventually resorted to another procedures where the averaging is introduced at an earlier level of the equations of motion. This leads then to a program based on the optical potential concept which introduces further physical simplifications, such as e. g. the neglect of higher order correlations implicit in the low density expansion. It is believed that the optical model approach [2, 11, 12] provides the most effective framework for calculating hadron scattering for medium to heavy nuclear targets. On the other hand, procedures based on the multiple scattering equations, free from any extra assumptions, seem to be quite feasible for light nuclei. Recently, such calculations have indeed been accomplished [13] by using the Monte Carlo techniques to perform the necessary integrations. Since the latter approach is not restricted to small angles scattering, it should provide at high energies a natural extension of the Glauber approximation. At very high energies, however, where for practical reasons one cannot utilize the partial wave expansion of the two body scattering amplitudes, the direct solution of the underlying equations becomes prohibitively difficult even for the lightest targets.

In this paper we wish to return to the problem of the scattering from N non-overlapping potentials. We choose not to work with appropriate Schrödinger equation, using instead the multiple scattering equations as formulated by Watson [12]. In these equations the microscopic potentials do not occur at all as they have been eliminated at the onset in favour to the corresponding two body on-shell amplitudes and the resulting equations may be looked at as self-consistency conditions for the amplitudes. Thus, our approach parallels the works by Gibbs and co-workers [13] but we differ from them in that we have lifted the so-called pole approximation introduced by the latter authors in a somewhat heuristic manner. Our goal is to formulate the multiple scattering problem in such a way as to obtain a solvable scheme for any two body amplitudes, including the

important case when the partial wave expansion cannot be exploited. And yet our ultimate equations will turn out to be free from integrations connected with the propagation between two successive scatterings.

In Section 2 we introduce a differential operator D which is then shown to factorize the Green funciton. Next, we consider a two body scattering problem and express the wave function in the space outside the interaction volume in terms of D and the on-shell scattering amplitude. In Section 3 we turn to the scattering from N non-overlapping centres and review briefly the multiple scattering formalism. We then use again the factorized form of the Green function and derive our basic set of equations for the amplitudes. We demonstrate that for the case where the two body scattering amplitudes contain a few partial waves, the problem reduces to solving a set of linear algebraic equations. Anticipating future applications, in Section 4 we consider a two body target for which explicit closed form solutions can be produced. In Section 5 we return again to the general N-body case and discuss various approximate procedures for solving the multiple scattering equations. We present two non-eikonal schemes which do not require partial wave expansions and show that our framework amalgamates the results obtained by other authors. Finally, Section 6 contains our conclusions.

2. *The differential operator D*

To begin the development at the simplest possible point, and one that will prove later quite useful, we shall consider scattering by a single central potential $V(r)$. We shall assume that $V(r)$ is localized in range, i. e. vanishes outside a sphere of a radius R. The wave function in this outer space is, of course, completely determined by the corresponding on-shell scattering amplitude. Our problem is the following: given the on-shell amplitude, construct the wave function. We should add immediately that we want to evaluate the wave function not only in the asymptotic region, but also for small separations close to the interaction volume. Although the above problem is solved quite trivially by expanding the scattering amplitude in partial waves, we shall find an alternative solution which possessing the full utility of partial wave expansion will be also useful for high energies where the partial wave method may become somewhat impractical.

Our method is based on a suitable factorization of the free Green function. Consider then the familiar expansion of the Green function in spherical harmonics

$$\frac{e^{ik|r-r'|}}{|r-r'|} = ik \sum_{l=0}^{\infty} j_l(kr')h_l(kr)P_l(\boldsymbol{n}\cdot\boldsymbol{n}'), \qquad r > r', \tag{1}$$

where $\boldsymbol{n} = \boldsymbol{r}/r$, $\boldsymbol{n}' = \boldsymbol{r}/r$, $j_l(x)$ and $h_l(x)$ are, respectively, the spherical Bessel and Hankel functions as defined in Ref. [14]. We shall rearrange now the series (1) so as to obtain an expansion in powers of $(1/kr)$. This is achieved by introducing a differential operator $D(\boldsymbol{n})$ defined in the following way

$$D(\boldsymbol{n}) = 1 + \sum_{\lambda=1}^{\infty} \frac{\boldsymbol{L}^2(\boldsymbol{L}^2-1\cdot 2)(\boldsymbol{L}^2-2\cdot 3)\ldots[\boldsymbol{L}^2-\lambda(\lambda+1)]}{\lambda!(-2ikr)^{\lambda}}, \tag{2}$$

where $\boldsymbol{L}$ is the angular momentum operator

$$\boldsymbol{L} = -i\boldsymbol{n}\times\frac{\partial}{\partial \boldsymbol{n}} \tag{3}$$

whose square is thus given by the expression

$$\boldsymbol{L}^2 = (n_i n_j - \delta_{ij})\frac{\partial}{\partial n_i}\frac{\partial}{\partial n_j} + 2n_i\frac{\partial}{\partial n_i}. \tag{4}$$

As seen directly from (2) the operator $D(\boldsymbol{n})$ commutes with $\boldsymbol{L}$ so that $D(\boldsymbol{n})\ Y_{lm}(\boldsymbol{n}) = D_l\ Y_{lm}(\boldsymbol{n})$, where the eigenvalue D_l is trivially found from (2) by setting $\boldsymbol{L}^2 = l(l+1)$ in which case the summation terminates after l terms.

The operators under the sum in (2) act similarly as projection operators, i. e. the first term admits all waves higher than the s-wave, the second term, respestively, higher than the p-wave, etc. Consequently, when $D(\boldsymbol{n})$ is applied to a single partial wave $P_l(\boldsymbol{n}'\cdot\boldsymbol{n})$, the summation in (2) will, as before, terminate after l terms and one is left with a polynomial in $(1/kr)$. This polynomial is related to the $h_l(kr)$ function and one finds

$$(e^{ikr}/r)D(\boldsymbol{n})P_l(\boldsymbol{n}\cdot\boldsymbol{n}') = ki^{l+1}h_l(kr)P_l(\boldsymbol{n}\cdot\boldsymbol{n}'). \tag{5}$$

Inserting (5) in (1) one obtains a useful representation of the Green function

$$\frac{e^{ik|r-r'|}}{|\boldsymbol{r}-\boldsymbol{r}'|} = \frac{e^{ikr}}{r}D(\boldsymbol{n})e^{-ik\boldsymbol{n}\cdot\boldsymbol{r}'}, \qquad r > r'. \tag{6}$$

Notice that in the above formula the dependence on r and r' factorizes in such a way that spherical wave is expressed in terms of a plane wave. In order to find the wave function outside the interaction volume, we write the integral equation for $\Psi(\boldsymbol{r})$

$$\Psi(\boldsymbol{r}) = e^{ikr} - \frac{2\mu}{4\pi}\int\frac{e^{ik|r-r'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}V(r')\Psi(\boldsymbol{r}')d^3r', \tag{7}$$

where μ is the mass of the projectile. Using now formula (6), one has for $r > R$ (we put a superfix out to stress that)

$$\Psi^{\text{out}}(\boldsymbol{r}) = e^{ikr} + \frac{e^{ikr}}{r}D(\boldsymbol{n})\left[-\frac{2\mu}{4\pi}\int e^{-ik\boldsymbol{n}r'}V(r')\Psi(\boldsymbol{r}')d^3r'\right]. \tag{8}$$

The expression in the square bracket in (8) is identified as the scattering amplitude and one obtains finally a closed form expression for the wave function

$$\Psi^{\text{out}}(\boldsymbol{r}) = e^{ikr} + (e^{kr}/r)D(\boldsymbol{n})f(k\boldsymbol{n}, \boldsymbol{k}), \tag{9}$$

where $f(k\boldsymbol{n}, \boldsymbol{k})$ is the on-shell scattering amplitude. In the asymptotic region (wave zone) we can set $D(\boldsymbol{n}) \approx 1$ and the scattered wave has usual form of the spherical wave but close to the scatterer (near zone) the spherical wave will be strongly distorted as then higher powers of $(1/kr)$ may become dominant. For low energies when $f(\boldsymbol{k}', \boldsymbol{k})$ is given in terms of a certain number of partial waves, the operator $D(\boldsymbol{n})$ contains only a finite number of

terms and using (5) and (9) we recover the angular momentum representation form of $\Psi(\boldsymbol{r})$. For high energies the partial wave expansion requires an extremely large number of terms and it is more economical to fit the scattering amplitude, as inferred from experiment, by some conveniently chosen function of the scattering angle which exhibits the diffractive nature of the amplitude. Formula (9) provides means to evaluate $\Psi(\boldsymbol{r})$ also in that case. Since we clearly have then the condition $kR \gg 1$, the high powers of $(1/kr)$ in (2) will give little contribution and the summation in (2) can be truncated. Ultimately, this approximation can be successively improved by adding higher order terms until a desired accuracy is attained. At every stage evaluation of $\Psi(\boldsymbol{r})$ is straightforward and the subsequent corrections are obtained merely by differentiating the scattering amplitude as indicated by (2) and (4). Furthermore, the operator $D(\boldsymbol{n})$ remains also quite useful if we have to do with non-central forces, e. g. if the scattering amplitude depends additionally on the spin vector $\boldsymbol{S}$. All calculations can be carried through in the Cartesian basis. To give just one example, we consider the scattering of spin $\frac{1}{2}$ particles. The scattering operator contains then the spin non-flip and spin-flip terms and may be written in the form

$$f(\boldsymbol{k}', \boldsymbol{k}) + \boldsymbol{\sigma} \cdot \boldsymbol{L} g(\boldsymbol{k}', \boldsymbol{k}),$$

where $\boldsymbol{\sigma}$ is the Pauli spin vector. The wave function $\Psi(\boldsymbol{r})$ can be written in a form which is a natural generalization of (8)

$$\Psi^{\text{out}}(r) = \left\{ e^{ikr} + \frac{e^{ikr}}{r} [D(\boldsymbol{n}) f(k\boldsymbol{n}, \boldsymbol{k}) + \boldsymbol{\sigma} \cdot \boldsymbol{L} D(\boldsymbol{n}) g(k\boldsymbol{n}, \boldsymbol{k})] \right\} \chi,$$

where χ denotes a spin $\frac{1}{2}$ spinor.

Concluding this section we shall give an integral formula which will be extensively used in the following. If $\Psi(\boldsymbol{r})$ and $\Phi(\boldsymbol{r})$ denote two arbitrary scalar functions of $\boldsymbol{n}$, then the following relation holds

$$\int \Psi(\boldsymbol{r}) D(\boldsymbol{n}) \Phi(\boldsymbol{r}) d\Omega_n = \int \Phi(\boldsymbol{r}) D(\boldsymbol{n}) \Psi(\boldsymbol{r}) d\Omega_n, \tag{10}$$

which is trivially proved by expanding $\Psi(\boldsymbol{r})$ and $\Phi(\boldsymbol{r})$ in spherical harmonics and recalling that $Y_{lr}(\boldsymbol{n})$ are eigenfunctions of $D(\boldsymbol{n})$.

3. Multiple scattering equations

We shall consider now the scattering from an assembly of N particles(nucleons). The common idea behind most approximation schemes is to reduce the many body scattering problem to a series of two body collisions. A particularly useful approach employs the set of multiple scattering equations and we are going now to recapitulate the main points of this metod [12].

Let H_0 be the initial state Hamiltonian of the projectile and the target (K and H_T, respectively)

$$H_0 = K + H_T. \tag{11}$$

The corresponding Green function is

$$G = (E - H_0 + i\varepsilon)^{-1} \tag{12}$$

and the projectile-target interaction V can be written as a sum of all projectile-nucleon interactions $v_\alpha (\alpha = 1, 2, \ldots, N)$

$$V = \sum_{\alpha=1}^{N} v_\alpha. \tag{13}$$

The state vector Ψ is a solution of the Lippmann-Schwinger equation

$$\Psi = \Phi + GV\Psi, \tag{14}$$

where Φ represents the ingoing wave. In order to replace the single many body equation (14) by a set of multiple scattering equations, we introduce the scattering matrices in the nuclear medium t_α. They are defined as solutions of the equations

$$t_\alpha = v_\alpha + v_\alpha G t_\alpha, \tag{15}$$

and in general the t_α are many body operators because G, which occurs in (15), is a many body operator. We define also a set of auxiliary state vectors Ψ_α which will describe the effective waves impinging on each target particle α. The key feature here is that the incident effective wave Ψ_α must not contain the wavelet produced by particle α itself. The vectors Ψ_α are introduced formally by means of the equation

$$v_\alpha \Psi = t_\alpha \Psi_\alpha, \tag{16}$$

which is analogous to the relation $V\Psi = T\Phi$ familar from the two body scattering problem. Thus, Ψ_α plays the same role as the incident wave in a two body scattering.

The state vector Ψ can now be eliminated in favour of the Ψ_α quantities and it can be shown by a straightforward algebra that the single equation (14) is equivalent to a set of equations [12]

$$\Psi = \Phi + G \sum_\alpha t_\alpha \Psi_\alpha, \tag{17}$$

$$\Psi_\alpha = \Phi + G \sum_{\beta \neq \alpha} t_\beta \Psi_\beta, \tag{18}$$

and the scattering matrix from the full target is evaluated from

$$T_{fi} = \langle \Phi_f | \sum_\alpha t_\alpha | \Psi_\alpha \rangle. \tag{19}$$

The above formulation is exact and Eqs (18) still possess all the complexities of a many body problem. On the other hand, if the target nucleons are replaced by a collection of fixed potentials, major simplifications occur in the above scheme. The target Hamiltonian vanishes in such a case, and G reduces to the free Green function of the projectile. Obviously, the t_α become identical with the free scattering matrices deduced from two body scattering and may be used as input in order to solve the multiple scattering Eqs (18).

If the centres do not overlap, only on-shell t_α matrices are required, otherwise one has to provide their off-shell continuation.

We shall now rewrite Eqs (17) and (18) in position representation replacing the t_α by the corresponding scattering amplitudes $f_\alpha(k', k)$. Thus, for fixed potentials Eqs (17) and (18) take the following form

$$\Psi(r) = e^{ik\cdot r} + \int \frac{e^{ik|r-r'|}}{|r-r'|} \sum_\alpha f_\alpha(r'-x_\alpha, r''-x_\alpha)\Psi_\alpha(r'')d^3r'd^3r'', \tag{20}$$

$$\Psi_\alpha(r) = e^{ikr} + \int \frac{e^{ik|r-r'|}}{|r-r'|} \sum_{\beta\neq\alpha} f_\beta(r'-x_\beta, r''-x_\beta)\Psi_\beta(r'')d^3r'd^3r'', \tag{21}$$

where $x_\alpha(\alpha = 1, 2, \ldots, N)$ denote the position vectors of each potential center and the coordinate representation scattering amplitude is the Fourier transform of the $f_\alpha(k', k)$ scattering amplitude

$$f_\alpha(x, y) = (2\pi)^{-6} \int e^{-ik'x} f_\alpha(k', k) e^{iky} d^3k' d^3k. \tag{22}$$

The scattering amplitude from the full target is obtained from (19)

$$F(k', k) = \sum_\alpha \int e^{-ik'r'} f_\alpha(r'-x_\alpha, r''-x_\alpha)\Psi_\alpha(r'')d^3r'd^3r''. \tag{23}$$

This equation can be written in a form which will prove useful later on

$$F(k', k) = \sum_\alpha e^{-ik'x_\alpha} F_\alpha(k', k) e^{ikx_\alpha}, \tag{24}$$

where the quantity $F_\alpha(k', k)$ defined as

$$F_\alpha(k', k) = \int e^{-ik\cdot x} f_\alpha(x, y)\Psi_\alpha(y)d^3xd^3y \tag{25}$$

may be interpreted as partial scattering amplitude describing a sequence of collisions with all target particles in which particle α was the last one struck. We shall show in the following that for non-overlapping potentials the multiple scattering equations can be reduced to set of algebraic equations for the partial amplitudes $F_\alpha(k', k)$. In order to bring the integral Eqs (20) and (21) to a tractable form, we shall again factorize the Green function by means of the differential operator D introduced in the preceding section. For a system of N scattering centres, in addition to the ingoing plane wave, at every scatterer α there are also incident wavelets comming from all the remaining particles, i. e. form $(N-1)$ directions $(x_\beta - x_\alpha)$, $\beta \neq \alpha$. Accordingly, our previous representation of the Green function has to be slightly modified and the expansion will be now in terms of the inverse mutual separation between the scattering centres. For two centres x_1 and x_2, in analogy with (6), we have

$$\frac{e^{ik|r-r'|}}{|r-r'|} = \frac{e^{ik|x_1-x_2|}}{|x_1-x_2|} D(n_{12}) e^{ikn_{12}(r-x_1)} e^{-ikn_{12}(r'-x_2)}, \tag{26}$$

$$|x_1-x_2| > |r-x_1|, \quad |x_1-x_2| > |r'-x_2|,$$

where $\boldsymbol{n}_{12}$ is a unit vector in the direction $(\boldsymbol{x}_1-\boldsymbol{x}_2)$ and the operator D is given by (2) where r has to be replaced by $|\boldsymbol{x}_1-\boldsymbol{x}_2|$. Inserting formula (26) in (21) and taking into account (25), we obtain

$$\Psi_\alpha(\boldsymbol{r}) = e^{i\boldsymbol{k}\boldsymbol{r}} + \sum_{\beta\neq\alpha} \frac{e^{ik|\boldsymbol{x}_\alpha-\boldsymbol{x}_\beta|}}{|\boldsymbol{x}_\alpha-\boldsymbol{x}_\beta|} D(\boldsymbol{n}_{\alpha\beta}) e^{ik\boldsymbol{n}_{\alpha\beta}(\boldsymbol{r}-\boldsymbol{x}_\alpha)} F_\beta(k\boldsymbol{n}_{\alpha\beta}, \boldsymbol{k}) e^{i\boldsymbol{k}\boldsymbol{x}_\beta}, \tag{27}$$

where $D(\boldsymbol{n}_{\alpha\beta})$ depends now on $|\boldsymbol{x}_\alpha-\boldsymbol{x}_\beta|$. When these mutual separations are large in comparison with the wavelength of the incident particle, $D(\boldsymbol{n}_{\alpha\beta})$ can be replaced by one, and formula (27) has a very simple interpretation. The effective wave Ψ_α is composed of the incident plane wave plus a superposition of the scattered plane waves emitted by all the remaining centres. If the separation $(\boldsymbol{x}_\alpha-\boldsymbol{x}_\beta)$ should not be regarded as large, then the reflected waves would become distorted and at small distances might differ appreciably from plane waves. The latter distortion is taken care of by the operator D. However, the effective wave Ψ_α, as given by formula (27), is not yet fully determined because we have not specified the partial amplitudes F_β occurring in this expression. As we have already mentioned the knowledge of the partial amplitudes $F_\beta(\boldsymbol{k}', \boldsymbol{k})$ solves our scattering problem completely. To derive our basic set of equations for the partial amplitudes F_α, we simply multiply (27) $e^{-i\boldsymbol{k}'(\boldsymbol{r}'-\boldsymbol{x})_\alpha} f_\alpha(\boldsymbol{r}'-\boldsymbol{x}_\alpha, \boldsymbol{r}-\boldsymbol{x}_\alpha)$ and integrate over $\boldsymbol{r}'$ and $\boldsymbol{r}$. Thus, one has

$$F_\alpha(\boldsymbol{k}', \boldsymbol{k}) e^{i\boldsymbol{k}\boldsymbol{x}_\alpha} = f_\alpha(\boldsymbol{k}', \boldsymbol{k}) e^{i\boldsymbol{k}\boldsymbol{x}_\alpha}$$

$$+ \sum_{\beta\neq\alpha} \frac{e^{ik|\boldsymbol{x}_\alpha-\boldsymbol{x}_\beta|}}{|\boldsymbol{x}_\alpha-\boldsymbol{x}_\beta|} D(\boldsymbol{n}_{\alpha\beta}) f_\alpha(\boldsymbol{k}', k\boldsymbol{n}_{\alpha\beta}) F_\beta(k\boldsymbol{n}_{\alpha\beta}, \boldsymbol{k}) e^{i\boldsymbol{k}\boldsymbol{x}_\beta}. \tag{28}$$

Since in Eq. (28) the vector $\boldsymbol{k}'$ is arbitrary, in order to solve (28) we may set $\boldsymbol{k}' = k\boldsymbol{n}_{\gamma\alpha}$ where $\gamma \neq \alpha$, obtaining a set of $N(N-1)$ equations for the $N(N-1)$ unknown amplitudes $F_\alpha(k\boldsymbol{n}_{\gamma\alpha}, \boldsymbol{k})$

$$F_\alpha(k\boldsymbol{n}_{\gamma\alpha}, \boldsymbol{k}) = f_\alpha(k\boldsymbol{n}_{\gamma\alpha}, \boldsymbol{k}) + \sum_{\beta\neq\alpha} \frac{e^{ikd_{\alpha\beta}}}{d_{\alpha\beta}} e^{-ikd_{\alpha\beta}} [D(\boldsymbol{v}) f_\alpha(kn_{\gamma\alpha}, k\boldsymbol{v}) F_\beta(k\boldsymbol{v}, \boldsymbol{k})]_{\boldsymbol{v}=\boldsymbol{n}_{\alpha\beta}}, \tag{29}$$

where $\boldsymbol{d}_{\alpha\beta} = \boldsymbol{x}_\alpha-\boldsymbol{x}_\beta$, $\alpha, \gamma = 1, 2, 3, \ldots, N$, $\alpha \neq \gamma$. The set of equations (29) completes our scheme. The amplitudes $F_\alpha(\boldsymbol{k}', \boldsymbol{k})$ calculated from (28) and (29) when inserted in (24) give the ultimate expression for the scattering amplitude from the full target.

It should be emphasized that by transforming (21) into (28) we have managed to get rid of the integrations connected with the free propagation between two successive collisions. Consequently, we have no longer to deal with a system of integral equations, the resulting Eqs (28) being much simpler as they contain only differentiations (via D). A crude approximate solution of (28) can be obtained without calculations by dropping the second term on the right hand side of (28). This procedure yields the first order impulse approximation. The presence of the second term is connected with rescattering and since it depends on F_β

the system of Eqs (28) may be looked at as self-consistency conditions for the partial amplitudes $F_\alpha(\boldsymbol{k}', \boldsymbol{k})$.

We shall consider in some detail the solutions of (28) in two extreme situations: (*i*) low energy scattering where the amplitudes $f_\alpha(\boldsymbol{k}', \boldsymbol{k})$ are given in terms of a relatively small number of partial waves; and, (*ii*) the high energy scattering where so many partial waves contributes that the partial wave expansion has to be abandoned. Only in the former case the system of Eqs (28) can be solved exactly, whereas the high energy scattering requires approximate prodecures. Ultimately, one may think about an intermediate energy region where in principle both methods should be applicable and are bound to give the same answer. As the energy of the projectile increases the low energy method becomes gradually very laborious and less economical.

We shall conclude this section by describing the low energy method whereas the approximate procedures will be discussed in Section 5. Thus, we assume that the partial wave expansion of $f_\alpha(\boldsymbol{k}', \boldsymbol{k})$ terminates at some $l_{\max}$ value of the orbital momentum. A general solution to the problem of scattering from N non-overlapping potentials in the angular momentum basis has been given in Ref. [10] and in Section 5 we shall rederive that result by expanding our multiple scattering equations in spherical harmonics. Here, however, we would like to give an alternative solution which employs Cartesian tensors. Owing to the special form of the operator $D(\boldsymbol{n})$ given in (2), the indicated differentiations may be carried through explicitly making the Cartesian representation well suited for our purposes. Furthermore, additional degrees of freedom such as spin, or isospin can be also easily handled that way.

Given a unit vector $\hat{\boldsymbol{k}}$. one can construct symmetric in all indices traceless Cartesian tensors $t_{ij\ldots w}$ (e. g. $t_i = k_i$, $t_{ij} = k_i k_j - \frac{1}{3}\delta_{ij}$, etc.), so that tensors with l indices transform under the irreducible representation of the rotation group of weight l. Thus, the tensor of rank l replaces the spherical harmonics $Y_{lm}(\hat{\boldsymbol{k}})$ and one has

$$\underbrace{\boldsymbol{L}^2 t_{ij\ldots w}}_{l} = l(l+1)t_{ij\ldots w}. \tag{30}$$

Denoting by $t'_{ij\ldots w}$ similar tensors composed of the unit vector $\hat{\boldsymbol{k}}'$ (in the direction of the final momentum) the amplitude $f_\alpha(\boldsymbol{k}', \boldsymbol{k})$ expanded in partial waves takes the form

$$f_\alpha(\boldsymbol{k}', \boldsymbol{k}) = a^{(\alpha)} + 3b^{(\alpha)} t'_i t_i + \tfrac{1}{2}\cdot 5\, c^{(\alpha)} t'_{ij} t_{ij} + \ldots, \tag{31}$$

where $a^{(\alpha)}, b^{(\alpha)}, c^{(\alpha)}, \ldots$ are the usual $s, p, d, \ldots$ partial wave amplitudes, respectively. Inserting (31) in (28) we observe that the only dependence of $F_\alpha(\boldsymbol{k}', \boldsymbol{k})$ on $\boldsymbol{k}'$ comes from $f_\alpha(\boldsymbol{k}', \boldsymbol{k})$ whereas the second vector $\boldsymbol{k}$ entres in a more complicated way. Thus, the amplitude $F_\alpha(\boldsymbol{k}'\, \boldsymbol{k})$ has to have the following structure which exhibits the dependence on $\boldsymbol{k}'$

$$F_\alpha(\boldsymbol{k}', \boldsymbol{k}) = A^{(\alpha)} + t'_i B_i^{(\alpha)} + t'_{ij} c_{ij}^{(\alpha)} + \ldots, \tag{32}$$

where $A^{(\alpha)}, B_i^{(\alpha)}, C_{ij}^{(\alpha)}$ are as yet unknown scalar, vector, tensor, etc. quantities, respectively. They depend on the vector $\boldsymbol{k}$ and the position vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N$ and will be

determined from Eqs (28). Substituting (31) and (32) into (28), one has

$$A^{(\alpha)}+\hat{\boldsymbol{k}}'\cdot\boldsymbol{B}^{(\alpha)}+\ \ldots\ = a^{(\alpha)}+3b^{(\alpha)}\hat{\boldsymbol{k}}'\cdot\hat{\boldsymbol{k}}+\ \ldots$$

$$+\sum_{\beta\neq\alpha}\frac{e^{ikd_{\alpha\beta}}}{d_{\alpha\beta}}e^{-ikd_{\alpha\beta}}D(\boldsymbol{n}_{\alpha\beta})\,[a^{(\alpha)}+3b^{(\alpha)}\hat{\boldsymbol{k}}'\boldsymbol{n}_{\alpha\beta}+\ \ldots]\,[A^{(\alpha)}+\boldsymbol{n}_{\alpha\beta}\boldsymbol{B}^{(\beta)}+\ \ldots]. \tag{33}$$

Using (2) and (30) the implied differentiations are easily carried through, viz.

$$D(\boldsymbol{n})n_i = D_1(kd)n_i,$$

$$D(\boldsymbol{n})n_in_j = D_2(kd)n_in_j+\tfrac{1}{3}\,[1-D_2(kd)]\delta_{ij}\ \text{etc},$$

where $D_1(x) = 1+i/x$; $D_2(x) = 1+3i/x-3/x^2$ etc. Comparing expressions multiplying the different irreducible tensors $t'_{ij\ldots w}$ on both sides of the Eq. (33), we obtain a set of equations for the $A^{(\alpha)}$, $\boldsymbol{B}^{(\alpha)}$, ... etc. quantities

$$A^{(\alpha)} = a^{(\alpha)}\left\{1+\sum_{\beta\neq\alpha}\frac{e^{ikd_{\alpha\beta}}}{d_{\alpha\beta}}e^{-ikd_{\alpha\beta}}[A^{(\beta)}+D_1(kd_{\alpha\beta})\boldsymbol{n}_{\alpha\beta}\boldsymbol{B}^{(\beta)}]\right\}, \tag{34a}$$

$$\boldsymbol{B}^{(\alpha)} = 3b^{(\alpha)}\left\{\hat{\boldsymbol{k}}+\sum_{\beta\neq\alpha}\frac{e^{ikd_{\alpha\beta}}}{d_{\alpha\beta}}e^{-ikd_{\alpha\beta}}[(D_1(kd_{\alpha\beta})A^{(\beta)}+D_2(kd_{\alpha\beta})\boldsymbol{n}_{\alpha\beta}\boldsymbol{B}^{(\beta)})\boldsymbol{n}_{\alpha\beta}\right.$$

$$\left.+\tfrac{1}{3}\,(1-D_2(kd_{\alpha\beta}))\boldsymbol{B}^{(\beta)}]\right\}, \tag{34b}$$

where in the above expressions for simplicity reasons we have retained only s and p waves. In order to solve Eqs (34) the vector $\boldsymbol{B}^{(\alpha)}$ is decomposed into three orthogonal components

$$\boldsymbol{B}^{(\alpha)} = B_{\parallel}^{(\alpha)}\hat{\boldsymbol{k}}+B_{\perp}^{(\alpha)}\boldsymbol{n}_{\perp}+B_{\times}^{(\alpha)}\hat{\boldsymbol{k}}\times\boldsymbol{n}_{\perp},$$

where $\boldsymbol{n}_{\perp}$ is arbitrary unit vector perpendicular to the direction of incidence. Projecting out from (34b) the three components of $\boldsymbol{B}^{(\alpha)}$ we end up with three scalar equations which supplemented by (34a) complete the scheme. Indeed, we have $4N$ equations for $4N$ quantities $A^{(\alpha)}$, $B_{\parallel}^{(\alpha)}$, $B_{\perp}^{(\alpha)}$, $B_{\times}^{(\alpha)}$. Substituting the solutions $A^{(\alpha)}$, $\boldsymbol{B}^{(\alpha)}$ back into (32), and then the corresponding expression for $F_{\alpha}(\boldsymbol{k}', \boldsymbol{k})$ into (24), we finally obtain the complete amplitude from the full target. The indicated procedure results in an exact solution.

4. Example: a two body target

To get some feeling for the use of the general formalism in practice, we are going now to investigate the simplest possible system of two scatterers. The results obtained here will be used in Section 5 to develop high energy approximation applicable for an N-body target.

For $N = 2$ there is only one separation vector $\boldsymbol{r} = \boldsymbol{x}_1 - \boldsymbol{x}_2$ and the system of Eqs (28) takes the form

$$F_1(\boldsymbol{k}', \boldsymbol{k}) = f_1(\boldsymbol{k}', \boldsymbol{k}) + \frac{e^{ikr}}{r} e^{-ikr} D(\boldsymbol{n}) f_1(\boldsymbol{k}', k\boldsymbol{n}) F_2(k\boldsymbol{n}, \boldsymbol{k}), \tag{35a}$$

$$F_2(\boldsymbol{k}', \boldsymbol{k}) = f_2(\boldsymbol{k}', \boldsymbol{k}) + \frac{e^{ikr}}{r} e^{ikr} D(\boldsymbol{n}) f_2(\boldsymbol{k}', -k\boldsymbol{n}) F_1(-k\boldsymbol{n}, \boldsymbol{k}), \tag{35b}$$

where $\boldsymbol{n}$ is a unit vector along $\boldsymbol{r}$. The two equations for the functions $F_1(-k\boldsymbol{n}, \boldsymbol{k})$ and $F_2(k\boldsymbol{n}, \boldsymbol{k})$ are obtained from (29)

$$F_1(-k\boldsymbol{n}, \boldsymbol{k}) = f_1(-k\boldsymbol{n}, \boldsymbol{k}) + \frac{e^{ikr}}{r} e^{-ikr} [D(\boldsymbol{v}) f_1(-k\boldsymbol{n}, k\boldsymbol{v}) F_2(k\boldsymbol{v}, \boldsymbol{k})]_{\boldsymbol{v}=\boldsymbol{n}}, \tag{36a}$$

$$F_2(k\boldsymbol{n}, \boldsymbol{k}) = f_2(k\boldsymbol{n}, \boldsymbol{k}) + \frac{e^{ikr}}{r} e^{ikr} [D(\boldsymbol{v}) f_2(k\boldsymbol{n}, -k\boldsymbol{v}) F_1(-k\boldsymbol{v}, \boldsymbol{k})]_{\boldsymbol{v}=\boldsymbol{n}}, \tag{36b}$$

and, finally, the scattering amplitude from a two body target is

$$F(\boldsymbol{k}', \boldsymbol{k}) = e^{iqx_1} F_1(\boldsymbol{k}', \boldsymbol{k}) + e^{iqx_2} F_2(\boldsymbol{k}', \boldsymbol{k}), \tag{37}$$

where $\boldsymbol{q} = \boldsymbol{k} - \boldsymbol{k}'$. We shall now consider in some detail two extreme situations, low and high energy scattering, respectively.

(*i*) *Low energy scattering.* Eqs (34) for s and p wave scattering can be written as

$$A^{(1)} = a^{(1)} \left\{ 1 + \frac{e^{ikr}}{r} e^{-ikr} [A^{(2)} + D_2(kr) \boldsymbol{n} \cdot \boldsymbol{B}^{(2)}] \right\}, \tag{38a}$$

$$A^{(2)} = a^{(2)} \left\{ 1 + \frac{e^{ikr}}{r} e^{ikr} [A^{(1)} - D_2(kr) \boldsymbol{n} \cdot \boldsymbol{B}^{(1)}] \right\}, \tag{38b}$$

$$\boldsymbol{B}^{(1)} = 3b^{(1)} \left\{ \hat{\boldsymbol{k}} + \frac{e^{ikr}}{r} e^{-ikr} [D_1(kr) A^{(2)} \boldsymbol{n} + D_2(kr) (\boldsymbol{n} \cdot \boldsymbol{B}^{(2)}) \boldsymbol{n} + \tfrac{1}{3} (1 - D_2(kr)) \boldsymbol{B}^{(2)}] \right\}, \tag{38c}$$

$$\boldsymbol{B}^{(2)} = 3b^{(2)} \left\{ \hat{\boldsymbol{k}} + \frac{e^{ikr}}{r} e^{ikr} [-D_1(kr) A^{(1)} \boldsymbol{n} + D_2(kr) (\boldsymbol{n} \cdot \boldsymbol{B}^{(1)}) \boldsymbol{n} + \tfrac{1}{3} (1 - D_2(kr)) \boldsymbol{B}^{(1)}] \right\}. \tag{38d}$$

In order to find $A^{(\alpha)}$ and $\boldsymbol{B}^{(\alpha)}$ the two vector equations in (38) are multiplied by $\boldsymbol{n}$ and resulting scalar equations can be then solved for $A^{(1)}$, $A^{(2)}$, $\boldsymbol{B}^{(1)}\boldsymbol{n}$ and $\boldsymbol{B}^{(2)}\boldsymbol{n}$. Substituting the solutions again in (38) one finally obtains $\boldsymbol{B}^{(1)}$ and $\boldsymbol{B}^{(2)}$. Although this procedure is fairly simple, the explicit expressions for the two amplitudes F_α are rather lengthy and will not be given here. For purely s-wave, or purely p-wave two body scattering major simplification takes place and in either case our Eqs (38) reduce to those derived many years ago by Brueckner [6]. Particularly simple is the s-wave scattering as then one can set $D(\boldsymbol{n}) = 1$ and solve (36) directly for $F_1(-k\boldsymbol{n}, \boldsymbol{k})$ and $F_2(k\boldsymbol{n}, \boldsymbol{k})$ (we shall give explicit expressions later on, cf. (41)).

(*ii*) *High energy scattering — small angles approximation.* By assumption, this procedure accounts for no more than a single collision with each target constituent and no reflections are admitted in this scheme. For a two body target every triple collision already has to be regarded as a reflection because backward scattering necessarily takes place in such a case. This is seen also from (36) where the terms proportional to D contain backward scattering amplitudes. The small angles approximation neglects terms proportional to the backward scattering amplitudes, i. e. terms in the square bracket in (36). This approximation leads immediately to the following expressions for the partial amplitudes $F_\alpha(\boldsymbol{k}', \boldsymbol{k})$, $\alpha = 1, 2$,

$$F_1(\boldsymbol{k}', \boldsymbol{k}) = f_1(\boldsymbol{k}', \boldsymbol{k}) + \frac{e^{ikr}}{r} e^{-ikr} D(\boldsymbol{n}) f_1(\boldsymbol{k}', k\boldsymbol{n}) f_2(k\boldsymbol{n}, \boldsymbol{k}), \tag{39a}$$

$$F_2(\boldsymbol{k}', \boldsymbol{k}) = f_2(\boldsymbol{k}', \boldsymbol{k}) + \frac{e^{ikr}}{r} e^{ikr} D(\boldsymbol{n}) f_2(\boldsymbol{k}', -k\boldsymbol{n}) f_1(-k\boldsymbol{n}, \boldsymbol{k}), \tag{39b}$$

where the amplitudes $f_\alpha(\boldsymbol{k}', \boldsymbol{k})$ may contain any number of partial waves. Further simplifications take place when formulae (39) are inserted in (37) and the resulting amplitude from the full target is averaged over the orientation of $\boldsymbol{n}$. The double scattering contribution, using (10) and (6), can be further reduced as follows

$$\frac{e^{ikr}}{r} \int e^{-iKr} D(\boldsymbol{n}) f_1(\boldsymbol{k}', k\boldsymbol{n}) f_2(k\boldsymbol{n}, \boldsymbol{k}) d\Omega_n = \frac{k}{r} \int \frac{e^{i|k\boldsymbol{n}-\boldsymbol{K}|r}}{|k\boldsymbol{n}-\boldsymbol{K}|} f_1(\boldsymbol{k}', k\boldsymbol{n}) f_2(k\boldsymbol{n}, \boldsymbol{k}) d\Omega_n, \tag{40}$$

where $\boldsymbol{K} = \frac{1}{2}(\boldsymbol{k}+\boldsymbol{k}')$. Introducing formally a two dimensional momentum transfer $\boldsymbol{Q} = k\boldsymbol{n}-\boldsymbol{K}$, the scattering amplitude averaged over $\boldsymbol{n}$ takes the form [15, 16]

$$\langle F(\boldsymbol{k}', \boldsymbol{k})\rangle_n = j_0(\tfrac{1}{2} qr)\,[f_1(\boldsymbol{k}', \boldsymbol{k}) + f_2(\boldsymbol{k}', \boldsymbol{k})]$$

$$+ \frac{1}{4\pi K} \int_{\text{ring}} \frac{e^{iQr}}{Qr} [f_1(\boldsymbol{k}', \boldsymbol{Q}+\boldsymbol{K}) f_2(\boldsymbol{Q}+\boldsymbol{K}, \boldsymbol{k}) + f_2(\boldsymbol{k}', \boldsymbol{Q}+\boldsymbol{K}) f_1(\boldsymbol{Q}+\boldsymbol{K}, \boldsymbol{k})] d^2Q, \tag{41}$$

where the radii of the ring are given by the inequality $|k-K| \leqslant Q \leqslant |k+K)$. In the extremely high energy limit, (41) reduces to the Glauber amplitude. Indeed, the ring can be extended to the whole plane, the off-shell part the propagation proportional to $\cos(Qr)/Qr)$ goes to zero [15, 16] for eikonalized amplitudes $f_\alpha(\boldsymbol{k}', \boldsymbol{k})$, while the on-shell part yields the Glauber shadow term [1]. Thus, multiplying (41) by the density $\varrho(r)$ and integrating over r, one eventually obtains the well known expression [1]

$$\langle F(\boldsymbol{k}', \boldsymbol{k})\rangle_r = R(\tfrac{1}{2} q) f_1(\boldsymbol{q}) + \frac{i}{4\pi k} \int S(Q) f_1(\boldsymbol{q}+\boldsymbol{Q}) f_2(\boldsymbol{q}-\boldsymbol{Q}) d^2Q + (1 \leftrightarrow 2), \tag{42}$$

where $S(q)$ is the formfactor associated with $\varrho(r)$. The effective expansion parameter in the above scheme is the ratio of the backward scattering amplitude to the (average) separa-

tion of the scattering centres. Another possible expansion parameter may be (1/kr as this quantity enters the operator D. The latter scheme will be called hereafter large separation approximation. Depending on the circumstances, one can develop a hybrid scheme where both $f(\pi)$ and $(1/kr)$ might be regarded as small.

(iii) High energy scattering — large separation approximation. This scheme is based on a systematic expansion in powers of (1/kr) whereas the backward scattering amplitudes are retained. Thus, reflections are included in contrast with the small angles approximation. The present approximation should be therefore adequate to study large angles scattering. The expansion of D has already been given in (2), it is sufficient then to expand the partial amplitudes $F_\alpha(\boldsymbol{k}', \boldsymbol{k})$ in powers of $(1/kr)$

$$F_\alpha(\boldsymbol{k}', \boldsymbol{k}) = F_\alpha^{(0)}(\boldsymbol{k}', \boldsymbol{k}) + \frac{1}{(-2ikr)} F_\alpha^{(1)}(\boldsymbol{k}', \boldsymbol{k}) + \frac{1}{(-2ikr)^2} F_\alpha^{(2)}(\boldsymbol{k}', \boldsymbol{k}) + \ldots, \tag{43}$$

where the functions $F_\alpha^{(0)}, F_\alpha^{(1)}, F_\alpha^{(2)}, \ldots$ will be evaluated iteratively. The zeroth order solution is obtained simply by setting $D = 1$ in (36). Thus, if only s-wave interaction is admitted, the zeroth order solution becomes exact. The solutions for $F_1^{(0)}(-k\boldsymbol{n}, \boldsymbol{k})$ and $F_2^{(0)}(k\boldsymbol{n}, \boldsymbol{k})$ are

$$F_1^{(0)}(-k\boldsymbol{n}, \boldsymbol{k}) = \left[f_1(-k\boldsymbol{n}, \boldsymbol{k}) + \frac{e^{ikr}}{r} e^{-ikr} f_1(\pi) f_2(k\boldsymbol{n}, \boldsymbol{k}) \right] \frac{1}{\Delta}, \tag{44a}$$

$$F_2^{(0)}(k\boldsymbol{n}, \boldsymbol{k}) = \left[f_2(k\boldsymbol{n}, \boldsymbol{k}) + \frac{e^{ikr}}{r} e^{ikr} f_2(\pi) f_1(-k\boldsymbol{n}, \boldsymbol{k}) \right] \frac{1}{\Delta}, \tag{44b}$$

where we have used abbreviated notation $f_\alpha(\pi)$ to denote the backward scattering amplitude, and $\Delta = 1 - f_1(\pi) f_3(\pi) (e^{ikr}/r)^2$. The corresponding zeroth order terms in (43) are

$$F_1^{(0)}(\boldsymbol{k}', \boldsymbol{k}) = f_1(\boldsymbol{k}', \boldsymbol{k}) + \frac{e^{ikr}}{r} e^{-ikr} f_1(\boldsymbol{k}', k\boldsymbol{n}) F_2^{(0)}(k\boldsymbol{n}, \boldsymbol{k}), \tag{45a}$$

$$F_2^{(0)}(\boldsymbol{k}', \boldsymbol{k}) = f_2(\boldsymbol{k}', \boldsymbol{k}) + \frac{e^{ikr}}{r} e^{ikr} f_2(k', -k\boldsymbol{n}) F_1^{(0)}(-k\boldsymbol{n}, \boldsymbol{k}). \tag{45b}$$

The above expressions inserted into (37) will give the zeroth order scattering amplitude from the two body target. The resulting formula coincides with the amplitude obtained in Ref. [8] using classical arguments. The essence of the above approximation is that the wave scattered from scatterer 1 can be approximated by a plane wave when it approaches the second scatterer and vice versa. The separation r must be therefore much bigger than the wavelength of the projectile. The denominator Δ describes reflections, indeed when Δ is expanded in powers of the backward scattering amplitudes, each term of the expansion may be identified with a multiple scattering process in which the projectile oscillates between 1 and 2 several times. If only s-wave contributes, formulae (45) give the exact scattering amplitudes from two fixed centres well known in the literature [6, 10]. Inserting the ex-

pressions (43) and (2), correct up to terms of the order of $(1/kr)$ in (36), one obtains two algebraic equations for $F_1^{(1)}(-k\boldsymbol{n}, \boldsymbol{k})$ and $F_2^{(1)}(k\boldsymbol{n}, \boldsymbol{k})$. The first order solutions are

$$F_1^{(1)}(-k\boldsymbol{n}, \boldsymbol{k}) = \frac{e^{ikr}}{r} \frac{1}{\Delta} \left\{ e^{-ikr} [\boldsymbol{L}^2 f_1(-k\boldsymbol{v}, k\boldsymbol{n}) F_2^{(0)}(k\boldsymbol{n}, \boldsymbol{k})]_{v=n} \right.$$

$$\left. + \frac{e^{ikr}}{r} f_1(\pi) [\boldsymbol{L}^2 f_2(k\boldsymbol{v}, -k\boldsymbol{n}) F_1^{(0)}(-k\boldsymbol{n}, \boldsymbol{k})]_{v=n} \right\}, \tag{46a}$$

$$F_2^{(1)}(k\boldsymbol{n}, \boldsymbol{k}) = \frac{e^{ikr}}{r} \frac{1}{\Delta} \left\{ e^{ikr} [\boldsymbol{L}^2 f_2(k\boldsymbol{v}, -k\boldsymbol{n}) F_1^{(0)}(-k\boldsymbol{n}, \boldsymbol{k})]_{v=n} \right.$$

$$\left. + \frac{e^{ikr}}{r} f_2(\pi) [\boldsymbol{L}^2 f_1(-k\boldsymbol{v}, k\boldsymbol{n}) F_2^{(0)}(k\boldsymbol{n}, \boldsymbol{k})]_{v=n} \right\}, \tag{46b}$$

and similarly one can evaluate the second, third, ... etc. orders. The second order solution becomes exact if only s and p waves contribute, and generally if $f_\alpha(\boldsymbol{k}', \boldsymbol{k})$ is given in terms of $l_{\max}$ partial waves then the solution accurate up to terms of the order $(1/kr)^{2l_{\max}}$ gives automatically the exact solution. Thus, the low energy method and the systematically improved large separation approximation lead to the same result.

5. *Applications and approximations*

In this section we shall discuss various applications of the general formalism developed in Section 3. We shall also show that the different approaches considered in the literature follow directl from the multiple scattering equations (28).

a. The Watson series

The explicit expression for the Watson series [12] is obtained by iterating Eq. (28). The scattering amplitude from the full target has the following form

$$F(\boldsymbol{k}', \boldsymbol{k}) = \sum_\alpha e^{-i\boldsymbol{k}'\boldsymbol{x}_\alpha} f_\alpha(\boldsymbol{k}', \boldsymbol{k}) e^{i\boldsymbol{k}\boldsymbol{x}_\alpha}$$

$$+ \sum_{\alpha,\beta(\alpha\neq\beta)} e^{-i\boldsymbol{k}'\boldsymbol{x}_\alpha} \frac{e^{ikd_{\alpha\beta}}}{d_{\alpha\beta}} D(\boldsymbol{n}_{\alpha\beta}) f_\alpha(\boldsymbol{k}', k\boldsymbol{n}_{\alpha\beta}) f_\beta(k\boldsymbol{n}_{\alpha\beta}, \boldsymbol{k}) e^{i\boldsymbol{k}\boldsymbol{x}_\beta}$$

$$+ \sum_{\substack{\alpha,\beta,\gamma \\ (\alpha\neq\beta,\beta\neq\gamma)}} e^{-i\boldsymbol{k}'\boldsymbol{x}_\alpha} \frac{e^{ikd_{\alpha\beta}}}{d_{\alpha\beta}} \frac{e^{ikd_{\beta\gamma}}}{d_{\beta\gamma}} D(\boldsymbol{n}_{\alpha\beta}) D(\boldsymbol{n}_{\beta\gamma}) f_\alpha(\boldsymbol{k}', k\boldsymbol{n}_{\alpha\beta}) f_\beta(k\boldsymbol{n}_{\alpha\beta}, k\boldsymbol{n}_{\beta\gamma}) f_\gamma(k\boldsymbol{n}_{\beta\gamma}, \boldsymbol{k}) e^{i\boldsymbol{k}\boldsymbol{x}_\gamma} + \ldots, \tag{47}$$

where $\boldsymbol{d}_{\alpha\beta} = \boldsymbol{x}_\alpha - \boldsymbol{x}_\beta$, etc. We would like to emphasize that even though the above expression contains the operators D, still each term of this series may be brought to a managable form. For low energies the expansion (2) terminates after $(2l_{\max}+1)$ terms if $f_\alpha(\boldsymbol{k}', \boldsymbol{k})$ is

given in terms of $l_{\max}$ partial waves. For high energies one can introduce the large separation approximation and truncate the series (2) for D, as discussed in the preceding Section. Nevertheless, the Watson series (47) in general does not seem to have much importance as far as practical applications are concerned. This series is very slowly convergent, and in many cases of practical interest probably even divergent.

b. Partial wave basis

The general solution to the scattering problem from an assembly of non-overlapping potentials has been obtained in partial wave basis by Agassi and Gal [10] who considered the N-body Schrödinger equation. We shall show that their results follow immediately from the multiple scattering equations (28). To this end we expand both the scattering amplitude $f_\alpha(\boldsymbol{k}', \boldsymbol{k})$ and the partial amplitude $F_\alpha(\boldsymbol{k}', \boldsymbol{k})$ in spherical harmonics

$$f_\alpha(\boldsymbol{k}', \boldsymbol{k}) = 4\pi \sum_{l,m} f_l^{(\alpha)}(k) Y_{lm}(\hat{\boldsymbol{k}}') Y_{lm}^*(\hat{\boldsymbol{k}}), \tag{48}$$

$$F_\alpha(\boldsymbol{k}', \boldsymbol{k}) = \sum_{l,m} i^{-l} f_l^{(\alpha)}(k) b_{lm}^{(\alpha)}(\boldsymbol{k}) Y_{lm}(\hat{\boldsymbol{k}}'), \tag{49}$$

where $f_l^{(\alpha)}(k)$ are partial wave amplitudes associated with the corresponding phase shifts $\delta_{\alpha l}$ by

$$k f_l^{(\alpha)}(k) = e^{i\delta_{\alpha l}} \sin \delta_{\alpha l}. \tag{50}$$

In the expansion (49) the coefficients $b_{lm}^{(\alpha)}$ are as yet unknown and we are going to determine them from the multiple scattering equations for the amplitudes. To demonstrate the correspondence with the formalism of Agassi and Gal we follow their notation and introduce the matrix $F_{lm,l'm'}(\boldsymbol{d})$ defined by the formula

$$F_{lm,l'm'}(\boldsymbol{d}) = \frac{4\pi}{ik} \frac{e^{ikd}}{d} D(\hat{\boldsymbol{d}}) i^{l'-l} Y_{lm}^*(\hat{\boldsymbol{d}}) Y_{l'm'}(\hat{\boldsymbol{d}}). \tag{51}$$

Using (2) and the additional theorem for two spherical harmonics it can be easily checked that the above matrix coincides with that defined in Ref. [10]. Inserting the expansions (48) and (49) into (28) we obtain the following set of equations for the coefficients $b_{lm}^{(\alpha)}$

$$b_{lm}^{(\alpha)}(\boldsymbol{k}) = 4\pi i^l Y_{lm}^*(\hat{\boldsymbol{k}}) e^{i\boldsymbol{k}\boldsymbol{x}_\alpha} + ik \sum_{\substack{\beta,l'm' \\ (\beta \neq \alpha)}} F_{l'm',lm}(\boldsymbol{d}_{\alpha\beta}) f_{l'}^{(\beta)}(k) b_{l'm'}^{(\beta)}(\boldsymbol{k}). \tag{52}$$

The above equations are identical with Eqs (2.20) of Ref. [10]. As seen from (49), the knowledge of the coefficients $b_{lm}^{(\alpha)}(\boldsymbol{k})$ solves completely the scattering problem in partial wave basis. For completeness, we are giving also ultimate expression for the scattering amplitude from the full target

$$F(\boldsymbol{k}', \boldsymbol{k}) = \sum_{\alpha,l,m} e^{-i\boldsymbol{k}'\boldsymbol{x}_\alpha} i^{-l} f_l^{(\alpha)}(k) b_{lm}^{(\alpha)}(\boldsymbol{k}) Y_{lm}(\hat{\boldsymbol{k}}'). \tag{53}$$

c. Never-come-back approximation

For a two body target reflections are always connected with backward scattering, the projectile being forced to oscillate back and forth between the two scatterers. Therefore, for sufficiently high energies, where the backward scattering is suppressed, reflections should be rather unimportant. For targets composed of more than two scatterers the multiple scattering equations admit also more complicated reflections in which the projectile experiences a number of collisions with different scatterers and comes back eventually to the first scatterer closing thereby a full reflection cycle. Thus, for a multiparticle target not necessarily every reflection involves large angle scattering and it might have appeared that at least some reflections should be important, even at very high energy. Furthermore, it is not obvious from (29) how an approximation scheme could be devised which would allow to suppress large angles scattering. The difficulty stems from the fact that the mutual separations $d_{\alpha\beta}$ which enter Eqs (29) are rather inconvenient for our purposes because working with mutual separations we cannot distinguish between large and small angle scattering.

On the other hand, a high energy approximation scheme which neglects systematically backward scattering results quite naturally in the Jacobi coordinates. For an N-body target in the c. m. system ($\sum_{\alpha} \boldsymbol{x}_{\alpha} = 0$), we introduce $(N-1)$ coordinate vectors $\boldsymbol{\varrho}_1, \boldsymbol{\varrho}_2, \ldots, \boldsymbol{\varrho}_{N-1}$

$$\begin{aligned}
\boldsymbol{\varrho}_1 &= \boldsymbol{x}_1 - \boldsymbol{x}_2, \\
\boldsymbol{\varrho}_2 &= \tfrac{1}{2}(\boldsymbol{x}_1 + \boldsymbol{x}_2) - \boldsymbol{x}_3, \\
\boldsymbol{\varrho}_3 &= \tfrac{1}{3}(\boldsymbol{x}_1 + \boldsymbol{x}_2 + \boldsymbol{x}_3) - \boldsymbol{x}_4, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\boldsymbol{\varrho}_{N-1} &= \frac{1}{N-1}(\boldsymbol{x}_1 + \boldsymbol{x}_2 + \ldots + \boldsymbol{x}_{N-1}) - \boldsymbol{x}_N.
\end{aligned} \tag{54}$$

The philosophy now is again to replace the scattering from N centres by a series of two body collisions. Thus, the three body target might be looked at as composed of a two body subsystem plus the third body, the four body target, respectively might be decomposed into a three body subsystem plus the fourth body, etc. The above Jacobi coordinates are well suited to account for such a decomposition, and at every stage we have to deal with a two body problem. The high energy approximation scheme we are going to develop, neglects always the contribution due to backward scattering. This assumption simplifies the computations considerably and we obtain a hierarchy of nested two body scattering problems. Furthermore, it turns out that in these multiple scattering processes no target particle can be struck more than once in result of our approximation.

To obtain the scattering amplitude from an N-body target, we have to consider stepwise all the lighter targets composed of 2, 3, ... $(N-1)$ scatterers. We denote accordingly by $F^{(2)}(\boldsymbol{k}', \boldsymbol{k})$, $F^{(3)}(\boldsymbol{k}', \boldsymbol{k})$, ..., $F^{(N)}(\boldsymbol{k}', \boldsymbol{k})$ the scattering amplitudes from 2, 3, ..., N body targets, respectively. What we are now going to do, is to express the scattering amplitude $F^{(N)}(\boldsymbol{k}', \boldsymbol{k})$ in terms of $F^{(N-1)}(\boldsymbol{k}', \boldsymbol{k})$, which in turn is expressed in terms of $F^{(N-2)}(\boldsymbol{k}', \boldsymbol{k})$

etc., the nesting procedure being continued up to $F^{(2)}(\boldsymbol{k}', \boldsymbol{k})$, neglecting backward scattering contribution at every stage. Applying repeatedly formula (39), one is led to a set of recursive relations

$$F^{(2)}(\boldsymbol{k}', \boldsymbol{k}) = e^{i\frac{1}{2}q\varrho_1}\left[f_1(\boldsymbol{k}', \boldsymbol{k}) + \frac{e^{ik\varrho_1}}{\varrho_1} e^{-ik\varrho_1} D(\boldsymbol{n}_1) f_1(\boldsymbol{k}', k\boldsymbol{n}_1) f_2(k\boldsymbol{n}_1, \boldsymbol{k})\right]$$

$$+ e^{-i\frac{1}{2}q\varrho_1}\left[f_2(\boldsymbol{k}', \boldsymbol{k}) + \frac{e^{ik\varrho_1}}{\varrho_1} e^{ik\varrho_1} D(\boldsymbol{n}_1) f_2(\boldsymbol{k}', -k\boldsymbol{n}_1) f_1(-k\boldsymbol{n}_1, \boldsymbol{k})\right],$$

$$F^{(3)}(\boldsymbol{k}', \boldsymbol{k}) = e^{i\frac{1}{3}q\varrho_2}\left[F^{(2)}(\boldsymbol{k}', \boldsymbol{k}) + \frac{e^{ik\varrho_2}}{\varrho_2} e^{-ik\varrho_2} D(\boldsymbol{n}_2) F^{(2)}(\boldsymbol{k}', k\boldsymbol{n}_2) f_3(k\boldsymbol{n}_2, \boldsymbol{k})\right]$$

$$+ e^{-i\frac{2}{3}q\varrho_2}\left[f_3(\boldsymbol{k}', \boldsymbol{k}) + \frac{e^{ik\varrho_2}}{\varrho_2} e^{ik\varrho_2} D(\boldsymbol{n}_2) f_3(\boldsymbol{k}', -k\boldsymbol{n}_2) F^{(2)}(-k\boldsymbol{n}_2, \boldsymbol{k})\right],$$

. . . . . .

$$F^{(N)}(\boldsymbol{k}', \boldsymbol{k}) = e^{i\frac{1}{N}q\varrho_{N-1}}\left[F^{(N-1)}(\boldsymbol{k}', \boldsymbol{k}) + \frac{e^{ik\varrho_{N-1}}}{\varrho_{N-1}} e^{-ik\varrho_{N-1}} D(\boldsymbol{n}_{N-1})\right.$$

$$\left. F^{(N-1)}(\boldsymbol{k}', k\boldsymbol{n}_{N-1}) f_N(k\boldsymbol{n}_{N-1}, \boldsymbol{k})\right]$$

$$+ e^{-i\frac{N-1}{N}q\varrho_{N-1}}\left[f_N(\boldsymbol{k}', \boldsymbol{k}) + \frac{e^{ik\varrho_{N-1}}}{\varrho_{N-1}} e^{ik\varrho_{N-1}} D(\boldsymbol{n}_{N-1})\right.$$

$$\left. f_N(\boldsymbol{k}', -k\boldsymbol{n}_{N-1}) F^{(N-1)}(-k\boldsymbol{n}_{N-1}, \boldsymbol{k})\right], \tag{55}$$

where $\boldsymbol{n}_1, \boldsymbol{n}_2, \ldots, \boldsymbol{n}_{N-1}$ are unit vectors of $\varrho_1, \varrho_2, \ldots, \varrho_{N-1}$, respectively. It is evident from (55) that in the above scheme, in accordance with our previous assertion, no particle in the N-body target can be struck more than once. Indeed, on substitution of $F^{(2)}$ into $F^{(3)}$, then $F^{(3)}$ into $F^{(4)}$, etc. the resulting final formula for $F^{(N)}$ may be written as a sum of products of the different f_α. Now, we can go one step further and average over the orientation of the vectors $\boldsymbol{n}_\alpha$, $\alpha = 1, 2, \ldots, (N-1)$. Under the integrals one can get rid of the D operators by making use of (10) and (6), similarly as in Section 4. The procedure described in Section 4 can be actually extended even further and one may replace the solid angle integrations by the integrations over the corresponding two dimensional momentum transfer $\boldsymbol{Q}_\alpha$

$$\boldsymbol{Q}_\alpha = k\boldsymbol{n}_\alpha - \boldsymbol{K}_\alpha,$$

$$\boldsymbol{K}_\alpha = \frac{\alpha}{\alpha+1}\boldsymbol{k} + \frac{1}{\alpha+1}\boldsymbol{k}', \qquad \alpha = 1, 2, \ldots, (N-1).$$

The final expression for angle averaged scattering amplitude may be written in a recursive form

$$\langle F^{(N)}(\boldsymbol{k}', \boldsymbol{k})\rangle_{n_1,n_2,\ldots,n_{N-1}} = j_0\left(\frac{1}{N}\, q\varrho_{N-1}\right)\langle F^{(N-1)}(\boldsymbol{k}', \boldsymbol{k})\rangle_{n_1,n_2,\ldots,n_{N-2}}$$

$$+ j_0\left(\frac{N-1}{N}\, q\varrho_{N-1}\right) f_N(\boldsymbol{k}', \boldsymbol{k})$$

$$+ \frac{1}{4\pi K_{N-1}} \int \frac{e^{i\varrho_{N-1}Q_{N-1}}}{\varrho_{N-1}Q_{N-1}} \langle F^{(N-1)}(\boldsymbol{k}', \boldsymbol{Q}_{N-1}+\boldsymbol{K}_{N-1})\rangle_{n_1,n_2,\ldots,n_{N-2}} f_N(\boldsymbol{Q}_{N-1}+\boldsymbol{K}_{N-1}, \boldsymbol{k}) d^2Q_{N-1}$$

$$+ \frac{1}{4\pi K'_{N-1}} \int \frac{e^{i\varrho_{N-1}Q_{N-1}}}{\varrho_{N-1}Q_{N-1}} f_N(\boldsymbol{k}', \boldsymbol{Q}_{N-1}+\boldsymbol{K}'_{N-1})$$

$$\langle F^{(N-1)}(\boldsymbol{Q}_{N-1}+\boldsymbol{K}'_{N-1}, \boldsymbol{k})\rangle_{n_1,n_2,\ldots,n_{N-2}} d^2Q_{N-1}, \tag{56}$$

where $\boldsymbol{K}'_\alpha = \dfrac{\alpha}{\alpha+1}\,\boldsymbol{k}' + \dfrac{1}{\alpha}\,\boldsymbol{k}$. Of course, recurring down at the very bottom we will have the previously obtained amplitude $F^{(2)}(\boldsymbol{k}', \boldsymbol{k})$ given by (41) which in the present notation is

$$\langle F^{(2)}(\boldsymbol{k}', \boldsymbol{k})\rangle_{n_1} = j_0(\tfrac{1}{2}\, q\varrho_1)\,[f_1(\boldsymbol{k}', \boldsymbol{k}) + f_2(\boldsymbol{k}', \boldsymbol{k})]$$

$$+ \frac{1}{4\pi K_1} \int \frac{e^{i\varrho_1 Q_1}}{\varrho_1 Q_1}\,[f_1(\boldsymbol{k}', \boldsymbol{Q}_1+\boldsymbol{K}_1) f_2(\boldsymbol{Q}_1+\boldsymbol{K}_1, \boldsymbol{k})$$

$$+ f_2(\boldsymbol{k}', \boldsymbol{Q}_1+\boldsymbol{K}_1) f_2(\boldsymbol{Q}_1+\boldsymbol{K}_1, \boldsymbol{k})]\, d^2Q_1. \tag{57}$$

The amplitude (56) is completely equivalent to that obtained by Gurvitz et al. [16]. It has been emphasized by these authors, and we wish to restate it here, that the practical applications of the above scheme are possible only if the nuclear wave function depends on the Jacobi coordinates, or to be more precise, on the absolute values of the ϱ_α vectors.

Obviously, the presented above version of the never-come-back approximation is not the only one possible. Depending on the circumstances, it might be sometimes more appropriate, or more convenient to use a cluster model and decompose the target nucleus in two subsystems which then are regarded as composite and decomposed again, etc. Thus, the calculation may be performed at various levels of sophistication but according to the never-come-back approximation, one always neglects the contribution from backward scattering. The explicit form of the available nuclear wave function will certainly be an important factor deciding about proper decomposition of a given target.

d. Large separation approximation

For asymptotic energies the number of partial waves $l_{\max}$ necessary to represent the input amplitudes $f_\alpha(\boldsymbol{k}', \boldsymbol{k})$ rapidly increases ($l_{\max} \gg 1$) and the exact method of solving Eqs (28) becomes impractical. In this case it is more economical to resort to approximate

procedures, such as e. g. the just discussed never-come-back approximation. On the other hand, since the latter method applies only for small angles, we wish now to develop another procedure free from such restrictions. Denoting by R and d the range of the projectile-nucleon interaction and the average nucleon-nucleon separation, respectively, we have the condition $l_{\max} \sim kR \gg 1$, which in view of the non-overlap assumption ($d > 2R$) leads to the inequality $kd \gg 1$ making $(1/kd)$ a convenient expansion parameter of the theory. As a matter of fact, in the preceding section we have already employed an approximation scheme based on such an expansion in application to the two body target. Now, we shall discuss a more general N-body case.

The set of Eqs (29) may be rewritten symbolically in a matrix form, using a rather obvious notation

$$F = f + (G\hat{D}f)F, \tag{58}$$

where we have put a hat to stress that $\hat{D}$ is a matrix whose elements are differential operators. The restriction on the summation in (29) is taken care of by defining the matrix G as $[G]_{\alpha\beta} = \exp(ikd_{\alpha\beta} - i\boldsymbol{kd}_{\alpha\beta})/d_{\alpha\beta}$ if $\alpha \neq \beta$, and $[G_{\alpha\beta} = 0$ otherwise. Since we have $\frac{1}{2}N(N-1)$ different separations $d_{\alpha\beta}$, in order to facilitate the right counting of powers of $(kd_{\alpha\beta})^{-1}$ we introduce a formal expansion parameter λ which will be set $\lambda = 1$ in the final formulae (the expansions are in fact effected in powers of $(kd_{\alpha\beta})^{-1}$). Thus, let us formally expand F and $\hat{D}$ in powers of λ

$$F = F^{(0)} + \lambda F^{(1)} + \lambda^2 F^{(2)} + \ldots, \tag{59}$$

$$\hat{D} = I + \lambda \hat{D}^{(1)} + \lambda^2 \hat{D}^{(2)} + \ldots, \tag{60}$$

where I is a unit matrix and $\hat{D}^{(1)}$, $\hat{D}^{(2)}$, ... are obtained from (2) as the first, second, ... etc. terms in the expansion in powers of $(kr)^{-1}$. The $F^{(0)}, F^{(1)}$, ... etc. will be determined by an iterative procedure. Inserting the expansions (59) and (60) in (58) and comparing terms multiplying the same powers of λ we are left with the following set of linear equations for the amplitudes $F^{(0)}$, $F^{(1)}$, ... etc.:

$$\begin{aligned}
(I - Gf)F^{(0)} &= f, \\
(I - Gf)F^{(1)} &= (G\hat{D}^{(1)}f)F^{(0)}, \\
(I - Gf)F^{(2)} &= (G\hat{D}^{(2)}f)F^{(0)} + (G\hat{D}^{(1)}f)F^{(1)}, \\
(I - Gf)F^{(3)} &= (GD^{(3)}f)F^{(0)} + (G\hat{D}^{(2)}f)F^{(1)} + (G\hat{D}^{(1)}f)F^{(2)}, \text{ etc.}
\end{aligned} \tag{61}$$

As seen from (61), the righ-hand sides of the above equations can be always evaluated by carrying through the appropriate differentiations so that we have obtained a fully manageable iterative approximation scheme. The initial set of equations (29) has been again reduced to a system of algebraic equations (61) and by increasing the number of iterations the scattering amplitude can be evaluated with arbitrary accuracy. Although we have to do with a sequence of sets of equations, the palatable feature of (61) is that a single matrix inversion of $(I - Gf)$ is sufficient to obtain solutions for each set of equations.

e. Pole approximation

This approximation has been developed vigorously by Gibbs and collaborators [13]. To see how does it come about, let us note first that the multiple scattering equations (28) may be written in momentum space

$$F_\alpha(\boldsymbol{k}', \boldsymbol{k}) = f_\alpha(\boldsymbol{k}', \boldsymbol{k})$$

$$+e^{-i\boldsymbol{k}\boldsymbol{x}_\alpha}\sum_{\beta\neq\alpha}\frac{ik}{4\pi}\int f_\alpha(\boldsymbol{k}', k\boldsymbol{v})\Gamma(\boldsymbol{d}_{\alpha\beta}, \boldsymbol{v})F_\beta(k\boldsymbol{v}, \boldsymbol{k})d\Omega_v e^{i\boldsymbol{k}\boldsymbol{x}_\beta}, \tag{62}$$

where the function $\Gamma(\boldsymbol{d}, \boldsymbol{v})$ for non-overlapping potentials is

$$\Gamma(\boldsymbol{d}, \boldsymbol{v}) = \sum_{l=0}^{\infty} h_l(kd)i^l(2l+1)P_l(\boldsymbol{n}\boldsymbol{v}), \qquad \boldsymbol{n} = \boldsymbol{d}/d. \tag{63}$$

The above series defines a generalized function which using (5) may be transformed to the form

$$\Gamma(\boldsymbol{d}, \boldsymbol{v}) = (4\pi/ik)\,(e^{ikd}/d)D(\boldsymbol{n})\delta(\boldsymbol{n}-\boldsymbol{v}). \tag{64}$$

Inserting this expression into (62), the integration over $\boldsymbol{v}$ is carried through trivially and we recover out previous Eqs (28) deduced from configuration space considerations. The function $\Gamma(\boldsymbol{d}, \boldsymbol{v})$ results from integration of the Green's function over intermediate momenta

$$\Gamma(\boldsymbol{d}, \boldsymbol{v}) = \frac{2}{\pi ik}\int_0^\infty \frac{e^{ipd\boldsymbol{v}}p^2dp}{p^2-k^2-i_\varepsilon}. \tag{65}$$

The pole approximation of Gibbs [13] consist in completing in the above formula the countour of integration and retaining only the pole term which yields the simple result

$$\Gamma(\boldsymbol{d}, \boldsymbol{v}) \simeq \Gamma^{\text{pole}}(\boldsymbol{d}, \boldsymbol{v}) = 2\theta(\boldsymbol{v}\cdot\boldsymbol{d})e^{ik\boldsymbol{d}\cdot\boldsymbol{v}}. \tag{66}$$

It should be noted that this approximation differs from the purely on-shell propagation (ONS) where one neglects the principal value of the integral (65) with the result

$$\Gamma(\boldsymbol{d}, \boldsymbol{v}) \simeq \Gamma^{\text{ONS}}(\boldsymbol{d}, \boldsymbol{v}) = e^{ikd\boldsymbol{v}}. \tag{67}$$

The functions $\Gamma^{\text{Pole}}(\boldsymbol{d}, \boldsymbol{v})$ and $\Gamma^{\text{ONS}}(\boldsymbol{d}, \boldsymbol{v})$ given by (66) and (67), respectively, are, in contrast with (6), non-singular at $d = 0$. Both the on-shell and the off-shell (OFF) propagators can be written in terms of the D operator. The explicit forms are

$$\Gamma(\boldsymbol{d}, \boldsymbol{v}) = \Gamma^{\text{ONS}}(\boldsymbol{d}, \boldsymbol{v})+\Gamma^{\text{OFF}}(\boldsymbol{d}, \boldsymbol{v}), \tag{68}$$

$$\Gamma^{\text{ONS}}(\boldsymbol{d}, \boldsymbol{v}) = \frac{4\pi}{ik}\tfrac{1}{2}\left[\frac{e^{ikd}}{d}D(\boldsymbol{n})\delta(\boldsymbol{n}-\boldsymbol{v})-\frac{e^{-ikd}}{d}D^*(\boldsymbol{n})\delta(\boldsymbol{n}+\boldsymbol{v})\right], \tag{69}$$

$$\Gamma^{\text{OFF}}(\boldsymbol{d}, \boldsymbol{v}) = \frac{4\pi}{ik}\tfrac{1}{2}\left[\frac{e^{ikd}}{d}D(\boldsymbol{n})\delta(\boldsymbol{n}-\boldsymbol{v})+\frac{e^{ikd}}{d}D^*(\boldsymbol{n})\delta(\boldsymbol{n}+\boldsymbol{v})\right]. \tag{70}$$

The Green function in the pole approximation can be also cast into a form similar to (64) but the differential operator D has to be slightly modified. To obtain the relevant expression, $\Gamma^{\text{Pole}}(\boldsymbol{d}, \boldsymbol{v})$ is first expanded in Legendre polynomials

$$\Gamma^{\text{pole}}(\boldsymbol{d}, \boldsymbol{v}) = \sum_{l=0}^{\infty} f_l(kd) i^l (2l+1) P_l(\boldsymbol{n} \cdot \boldsymbol{v}), \tag{71}$$

where

$$f_l(x) = h_l(x) - i^{-l-1} \sum_{s=0}^{l} \frac{(l+s)!}{(l-s)!} \frac{1}{(s!)^2} \left(\frac{-1}{2}\right)^s \left(1 + i \frac{d}{dx}\right)^s \frac{1}{x}. \tag{72}$$

Defining now a new operator $D^{\text{Pole}}(\boldsymbol{n})$ as

$$D^{\text{pole}}(\boldsymbol{n}) = 1 + \sum_{\lambda=1}^{\infty} \frac{\boldsymbol{L}^2(\boldsymbol{L}^2 - 1 \cdot 2)(\boldsymbol{L}^2 - 2 \cdot 3) \ldots [\boldsymbol{L}^2 - \lambda(\lambda+1)]}{(-2ikd)^\lambda}$$

$$\times \left[\frac{1 - de^{-ikd}(\lambda!)^{-1}(ikd)^\lambda \left(1 + \frac{i}{k} \frac{\partial}{\partial d}\right)^\lambda \frac{1}{d}}{1 - e^{ikd}} \right], \tag{73}$$

the Green function $\Gamma^{\text{Pole}}(\boldsymbol{d}, \boldsymbol{v})$ is brought to the form

$$\Gamma^{\text{pole}}(\boldsymbol{d}, \boldsymbol{v}) = 4\pi e^{i \frac{1}{2} kd} \frac{\sin(\frac{1}{2} kd)}{(\frac{1}{2} kd)} D^{\text{pole}}(\boldsymbol{n}) \delta(\boldsymbol{n} - \boldsymbol{v}). \tag{74}$$

Formula (74) facilitates the comparison of $\Gamma^{\text{Pole}}(\boldsymbol{d}, \boldsymbol{v})$ with other forms of $\Gamma(\boldsymbol{d}, \boldsymbol{v})$ listed above.

6. *Conclusions*

We have considered the scattering from an assembly of fixed non-overlapping potentials what is supposed to simulate the scattering of high energy hadrons from nuclei. Since the underlying microscopic potentials are unknown we preferred not to work with the N-body Schrödinger equation. Our approach is based instead on a completely equivalent set of multiple scattering equations in which as an input appear the corresponding two body on-shell scattering amplitudes. Our basic result is the system of equations (28) for the partial amplitudes $F_\alpha(\boldsymbol{k}; \boldsymbol{k})$ which is free from the integrations connected with the propagation between two successive collisions. We have shown that the problem can be further simplified and reduces to solving a set of algebraic equations. If the two body scattering amplitudes can be represented in terms of a finite number of partial waves, the above reduction is exact. For asymptotic energies where the partial wave expansion is impractical, we have devised an iterative scheme which eventually leads again to a system of algebraic equations. Finally, we have demonstrated that our framework embraces

the other author's results. We believe that the proposed scheme will be quite adequate for calculations involving light nuclear targets. The systematic application of the large separation approximation which contains reflections to all orders and does not require partial wave expansion should be particularly useful for evaluating the scattering amplitude at large angles beyond the region of applicability of the Glauber approximation. Detailed calculations along these lines are in progress.

If the potentials overlap (and in the realistic case, to be sure, they do), the methods developed in this paper do not work. Since the off-shell scattering amplitudes are needed in that case there is not much one can do without this vital input information. In two extreme cases, however, very high and very low energies, one may devise approximate procedures to correct for the overlapping. Although the non-overlap method, strictly speaking is applicable for separations $d_{\alpha\beta} > 2R$, where R is the range of the microscopic potentials, but the non-overlap solution can be continued and regarded as approximately valid also for $d_{\alpha\beta} \lesssim 2R$. For very small separations (which means substantial overlap), the non-overlap approximation becomes rapidly unreliable and deviates strongly from the proper solution [3]. Indeed, as seen directly from (2), the high powers of $(1/r)$ make the non-overlap solution singular when $r = 0$. On the other hand, $d_{\alpha\beta} \to 0$ limit means that we have to do with a complete overlap so that the problem reduces to two body scattering and is again solvable, the corresponding solution being particularly simple for $E \to 0$ and $E \to \infty$. Thus, in principle, we are able to obtain the scattering amplitude for the concentric case ($d_{\alpha\beta} = 0$) and for the non-overlapping case ($d_{\alpha\beta} > 2R$), but the amplitude might be expected to be a smooth function of $d_{\alpha\beta}$ and one can interpolate between these two solutions [16].

Concluding, it is recognized that the adopted here FSA model also requires corrections resulting from the failure of closure, difference in kinematics, Fermi motion, etc. which have been completely ignored throughout the present paper but might appear non-negligible in a realistic calculation [17].

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