NONSTATIC CHARGED FLUID SPHERES IN GENERAL RELATIVITY

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A general method is described by which a class of exact solutions of Einstein's field equations is obtained representing nonstatic spherically symmetric distributions of charged perfect fluid. A class of static solutions is also obtained from the above as a special case. The method is a generalisation of that described earlier by Chakravarty et al. for nonstatic neutral fluid spheres. In addition to some previously known solutions, a new set of exact solutions is found. The dynamical behaviour of one of them (a generalisation of Nariai's solution for a neutral fluid sphere) is briefly discussed.

1. Introduction

Nonstatic solutions representing shear free motion of a sphere of charged perfect fluid having inhomogeneous density and pressure were discussed by several authors (Vaidya and Shah (1967), Faulkes (1969), Banerjee et al. (1975). Chakravarty et al. (1976) gave a method for deriving a class of exact solutions of Einstein's field equations representing shear free motion of neutral perfect fluid spheres. By this method they rediscovered some of the solutions found by earlier workers and also found a set of new solutions. We shall describe below (Section 2) a generalisation of this method which enables us to derive a class of solutions representing shearfree motion of charged spheres of perfect fluid. A class of static solutions is also obtained from the above as a special case. This method enables us to rederive some of the known solutions and in addition gives us a set of new solutions. In Section 3 we shall discuss the dynamical behaviour of one such solution, which is the charged analogue of the solution of a nonstatic sphere of neutral fluid discovered earlier by Nariai (1967).

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2. *Method of integrating the field equations*

Faulkes (1969) has studied the problem of charged spheres of perfect fluid in detail. So we shall not write down the detailed field equations here and only quote some of his results. Faulkes took the spherically symmetric line element in the isotropic form

$$ds^2 = e^{\nu}dt^2 - e^{\omega}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2), \tag{1}$$

where $\omega = \omega(r, t)$ and $\nu = \nu(r, t)$. Since the fluid is assumed to be perfect, the matter energy momentum tensor has the following components:

$$T_1^1 = T_2^2 = T_3^3 = -p, \quad T_4^4 = \varrho,$$

where p is the isotropic pressure and ϱ the mass density. Faulkes (1969b) then showed that the key equation in this case is

$$\frac{\partial^2 R}{\partial x^2} = f(x)R^2 + g(x)R^3, \tag{2}$$

where $x = r^2$ and $R = e^{-\omega/2}$, $f(x)$ and $g(x)$ being arbitrary functions of x alone. From this equation we obtain e^{ω}, then the equation corresponding to T_4^4 T_4^1 gives

$$e^{\nu/2} = \frac{c(t)}{\omega}, \tag{3}$$

where $c(t)$ is an arbitrary function of t. Thus the metric is completely determined. The remaining field equations yield the values of p, ϱ and the charge density σ.

We shall now describe a general method for deriving a class of solutions of Eq. (2). Let us write

$$R = (\xi + \theta)v(x), \tag{4}$$

where $\xi = \xi(r, t)$ and θ is a constant. Substituting this in (2) we obtain as first integral the equation

$$\xi'^2 = B(x)\xi^2(\xi^2 + a\xi + b), \tag{5}$$

where prime denotes derivative with respect to x and a, b are constants. Further θ is determined from the equation:

$$\theta(4\theta^2 - 3a\theta + 2b) = 0. \tag{6}$$

The solutions of this equation are

$$\theta = 0; \quad \theta = \theta_1 \equiv \frac{3a + (9a^2 - 32b)^{1/2}}{8}$$

or

$$\theta = \theta_2 \equiv \frac{3a - (9a^2 - 32b)^{1/2}}{8}. \tag{6a}$$

Further

$$B(x) = v^{-4}; \quad g(x) = 2v^{-6}; \quad f(x) = \tfrac{3}{2}(a-4\theta)v^{-5} \tag{7}$$

$v(x)$ is determined by the differential equation

$$v'^2 = K - h(\theta)v^{-2}, \tag{8}$$

where K is an arbitrary constant and

$$h(\theta) = 3a\theta - 6\theta^2 - b. \tag{9}$$

Equation (5) may now be integrated in the form

$$L = M + T, \tag{10}$$

where

$$L = \int \frac{d(1/\xi)}{(1+a/\xi+b/\xi^2)^{1/2}}, \quad M = \pm \int v^{-2}dx \tag{10a}$$

and T is an arbitrary function of time. We now proceed to examine the values of L and M in different cases. We first enumerate below the values of L in five different cases.

Case A, $$a = b = 0, \text{ then } L = \xi^{-1}. \tag{11}$$

Case B, $$a \neq 0,\ b = 0, \text{ then } L = \frac{2}{a}\sqrt{1+a/\xi}\,. \tag{12}$$

Case C, $$a \neq 0,\ a^2 = 4b, \text{ then } L = \frac{2}{a}\ln\left(1+\frac{a}{2\xi}\right). \tag{13}$$

Case D, $$a,\ b,\ a^2-4b \neq 0,\ b > 0, \text{ then } L = \frac{1}{\sqrt{b}}\ln\left[\frac{\sqrt{b}}{\xi}+\frac{a}{2\sqrt{b}}+\sqrt{1+a/\xi+b/\xi^2}\right]. \tag{14}$$

Case E, $$a \neq 0,\ b < 0, \text{ then } L = \frac{1}{\sqrt{-b}}\sin^{-1}\frac{2b/\xi+a}{\sqrt{a^2-4b}}\,. \tag{15}$$

Integration of (8), on the other hand, leads to the following different forms of v and M.

Case I, $$K = h = 0, \text{ then } v_{\mathrm{I}} = A \text{ (const.) and } M_{\mathrm{I}} = \pm A^{-2}x. \tag{16}$$

Case II, $$K > 0,\ h = 0, \text{ then } v_{\mathrm{II}} = \sqrt{K}\,x+l \text{ and } M_{\mathrm{II}} = \pm\frac{1}{\sqrt{K}\,(\sqrt{K}\,x+l)}, \tag{17}$$

where l is a constant of integration.

Case III, $$K = 0,\ h < 0, \text{ then } v_{\mathrm{III}} = (2\sqrt{-h}\,x+d)^{1/2} \text{ and}$$

$$M_{\mathrm{III}} = \pm\frac{1}{2\sqrt{-h}}\ln(2\sqrt{-h}\,x+d), \tag{18}$$

where d is a constant of integration.

Case IV, $\quad K;\ h \neq 0$, then $v_{IV} = (Kx^2+2nx+n^2/K+h/K)^{1/2}$, (19)

where n is an arbitrary constant. Two subcases may now be distinguished:

Case IVa, $\quad h < 0$, then $M_{IV_a} = \pm \dfrac{1}{2\sqrt{+h}} \ln \dfrac{Kx+n-\sqrt{-h}}{Kx+n+\sqrt{-h}}$; (20)

Case IVb, $\quad h > 0$, then $M_{IV_b} = \pm \dfrac{1}{\sqrt{+h}} \tan^{-1} \dfrac{Kx+n}{\sqrt{h}}$. (21)

Any allowed combination of the first set of cases $A-E$ with the second set I–IV will give us an exact solution representing a nonstatic charged fluid sphere by using Eqs (4) and (10). If, however, T is not an arbitrary function of t but a constant then we get the corresponding static solutions.

A_I: Eq. (6) gives $\theta = 0$. Hence from Eqs (4), (10), (11) and (16) we obtain

$$R = A(T \pm A^{-2}x)^{-1}. \tag{22}$$

This is identical with the solution obtained earlier by Faulkes (1969) in the special case when the total mass of the sphere vanishes but not the charge.

A_{II}: $$\theta = 0 \text{ and } R = \frac{v_{II}}{M_{II}+T}. \tag{23}$$

B_I: $$\theta = 0 \text{ and } R = \frac{4aA}{a^2(T \pm A^{-2}x)-4}. \tag{24}$$

This is identical with a special case of the solution representing a nonstatic charged fluid sphere found by Faulkes (1969b). This is the charged analogue of a neutral fluid sphere solution found earlier by Faulkes (1969a).

B_{II}: $$\theta = 0 \text{ and } R = \frac{4av_{II}}{a^2(M_{II}+T)^2-4}. \tag{25}$$

This is identical with a solution found by Banerjee et al. (1976), which is the charged analogue of a neutral fluid sphere solution found earlier by Banerjee and Banerji (1976).

B_{III}: $$\theta = \frac{3a}{4},\ h = -\frac{9a^2}{8} \text{ and } R = \left[\frac{4a}{a^2(M_{III}+T)^2-4} + \frac{3a}{4}\right] v_{III}. \tag{26}$$

B_{IVa}: $$\theta = \frac{3a}{4},\ h = -\frac{9a^2}{8} \text{ and } R = \left[\frac{4a}{a^2(M_{IVa}+T)^2-4} + \frac{3a}{4}\right] v_{IV}. \tag{27}$$

C_{III}: $$\theta = 0,\ h = -\frac{a^2}{4} \text{ and } R = \frac{av_{III}}{2[e^{(M_{III}+T)a/2}-1]}. \tag{28}$$

This is identical with a special case of the solution found by Vaidya and Shah (1967).

$$\theta = \theta_1 = a/2;\ h = -\frac{a^2}{4} \text{ and}$$

and

$$R = \frac{a}{2} v_{\rm III} \left[\frac{e^{(M_{\rm III}+T)a/2}}{e^{(M_{\rm III}+T)a/2}-1}\right]. \tag{29}$$

This is identical with a special case of the solution found by Banerjee et al. (1975).

$C_{\rm IVa}$:
$$\theta = 0,\ h = -a^2/4 \text{ and } R = \frac{a}{2} v_{\rm IV}\left[\frac{1}{e^{(M_{\rm IVa}+T)a/2}-1}\right], \tag{30}$$

$$\theta = a/2,\ h = -a^2/4 \text{ and } R = \frac{a v_{\rm IV}}{2}\left[\frac{e^{(M_{\rm IVa}+T)a/2}}{e^{(M_{\rm IVa}+T)a/2}-1}\right]. \tag{30a}$$

$C_{\rm IVb}$:
$$\theta = a/4,\ h = a^2/8 \text{ and } R = \frac{a v_{\rm IV}}{4}\left[\frac{e^{(M_{\rm IVb}+T)a/2}+1}{e^{(M_{\rm IVb}+T)a/2}-1}\right]. \tag{31}$$

$D_{\rm I}$:
$$\theta = 3a/8 = \sqrt{b/2} \text{ and } R = v_{\rm I}\left[\frac{2\sqrt{b}\, e^{\sqrt{b}(M_{\rm I}+T)}}{\left(e^{\sqrt{b}(M_{\rm I}+T)} - \dfrac{a}{2\sqrt{b}}\right)^2 - 1} + \frac{3a}{8}\right]. \tag{32}$$

$D_{\rm II}$:
$$\theta = 3a/8,\ R = v_{\rm II}\left[\frac{2\sqrt{b}\, e^{\sqrt{b}(M_{\rm II}+T)}}{\left(e^{\sqrt{b}(M_{\rm II}+T)} - \dfrac{a}{2\sqrt{b}}\right)^2 - 1} + \frac{3a}{8}\right]. \tag{33}$$

In order to save space we shall not give expressions for ξ. R can be obtained from them with the help of Eq. (4).

$D_{\rm III}$:
$$\theta = 0,\ h = -b \text{ and } \xi = \frac{2\sqrt{b}\, e^{\sqrt{b}(M_{\rm III}+T)}}{\left(e^{\sqrt{b}(M_{\rm III}+T)} - \dfrac{a}{2\sqrt{b}}\right)^2 - 1}. \tag{34}$$

This is the charged analogue of the neutral fluid solution found by Nariai (1967). We shall give a detailed discussion of the dynamical behaviour of this solution in Section 3.

$\theta = \theta_1$, $h = h(\theta_1)$, then ξ is given by (34). When $a > 0$, h is always negative but when $a < 0$ it is not always negative and only those negative values of a are to be considered for which $h < 0$.

$\theta = \theta_2$, $h = h(\theta_2)$ and ξ is given by (34). In this case h is always negative when $a < 0$ but not always when $a > 0$.

D_{IV}: In this case both positive and negative values of h are allowed. Hence there is no restriction on a.

$$\xi = \frac{2\sqrt{b}\, e^{(M_{IV}+T)}}{\left(e^{\sqrt{b}\,(M_{IV}+T)} - \dfrac{a}{2\sqrt{b}}\right)^2 - 1}. \tag{35}$$

E_{III}: $\theta = \theta_1 \neq 0$, $h = h(\theta_1)$ and $$\xi = \frac{-2b}{a - \sqrt{a^2-4b}\,\sin\{\sqrt{-b}\,(M_{III}+T)\}}; \tag{36}$$

$\theta = \theta_2 \neq 0$, $h = h(\theta_2)$ and ξ is given by (36).

E_{IVa}: $\theta = \theta_1$ or θ_2 and $$\xi = \frac{-2b}{a - \sqrt{a^2-4b}\,\sin\{\sqrt{-b}\,(M_{IVa}+T)\}}. \tag{37}$$

E_{IVb}: $\theta = \theta_0$ and $$\xi = \frac{-2b}{a - \sqrt{a^2-4b}\,\sin\{\sqrt{-b}\,(M_{IVb}+T)\}}. \tag{38}$$

Our results can be summarised in the form of a compact recipe for constructing exact solutions of the differential equation (2):

(1) Choose the values of the constants a and b arbitrarily. This choice determines its class A, B, ..., E. The choice of a, b determines $\theta+h$ from Eqs. (6) and (9).

(2) Now choose K arbitrarily and the values of h, K thus obtained determine the class I, II, ..., IVb.

3. *Dynamical behaviour of solution D_{III}*

Here

$$R = \xi v_{III} = \frac{2\sqrt{b}\, v_{III} e^{\sqrt{b}\,(M_{III}+T_1)}}{\left(e^{\sqrt{b}\,(M_{III}+T_1)} - \dfrac{a}{2\sqrt{b}}\right)^2 - 1}.$$

Let us write

$$T_1 = \pm\frac{1}{2\sqrt{-h}} \ln S_1^2, \quad S_1 = S_1(t),$$

$$M_{III}+T_1 = \pm\frac{1}{2\sqrt{-h}} \ln (ZS_1^2), \tag{39}$$

where $Z = (l+2\sqrt{-h r^2}) = 1+\gamma x$,
$l = 1$ and $2\sqrt{-h} = \gamma$,

$$R = \frac{2\sqrt{b}\, S_1 Z}{\left(S_1 Z^{1/2} - \dfrac{a}{2\sqrt{b}}\right)^2 - 1}. \tag{40}$$

Now replacing S_1 by $1/T$ and Z by y and introducing new constants which are related to the constants of Eq. (40), Eq. (40) simplifies to

$$R = \frac{T}{(a_1 T y^{-1/2} - \beta)^2 - \alpha}, \tag{41}$$

where a_1, β and α are constants, i. e.

$$e^{\omega/2} = \frac{(a_1 T y^{-1/2} - \beta)^2 - \alpha}{T}. \tag{41a}$$

If we set $\alpha = 0$, this reduces to the solution of Nariai. We shall now discuss the behaviour of the above solution. Using Eq. (3) we obtain $e^{\nu/2}$. If we put $C = 2\dot{T}/T$, we obtain

$$e^{\nu/2} = \frac{a_1^2 T^2/y - \beta^2 + \alpha}{(a_1 T y^{-1/2} - \beta)^2 - \alpha} = \frac{Z}{S}, \tag{42}$$

where

$$S = (a_1 T y^{-1/2} - \beta)^2 - \alpha, \tag{43}$$

$$Z = a_1^2 T^2/y - \beta^2 + \alpha. \tag{44}$$

We can then calculate the pressure p, matter density ϱ and the charge density σ from the field equations,

$$8\pi p = -\frac{4\gamma a_1^2 T^4}{S^2 y^3 Z} - \frac{2\ddot{T}}{T} \cdot \frac{S}{Z} + \frac{2\dot{T}^2}{T^2} \cdot \frac{S}{Z} - \frac{3\dot{T}^2}{T^2}, \tag{45}$$

$$8\pi\varrho = \frac{3\dot{T}^2}{T^2} + \frac{12 a_1 \gamma T^3}{S^3 y^{5/2}} (S+\alpha)^{1/2}, \tag{46}$$

$$4\pi\sigma = \pm \frac{3 a_1 \gamma \alpha^{1/2}}{\pi^{1/2} y^{5/2}} T^3/S^3. \tag{47}$$

It is evident from (47) that the constant α can take positive values only and $\alpha = 0$ corresponds to the case of uncharged fluid sphere discussed earlier by Nariai. We can also write the total charge up to the co-moving radius as (Bekenstein (1971))

$$q(r) = 4\pi \int_0^r e^{3\omega/2} \sigma r^2 dr = \pm \frac{a_1 \gamma \alpha^{1/2} r^3}{\pi^{1/2} y^{3/2}}. \tag{48}$$

Hence at the boundary $r = r_0$, $q(r_0)$ gives the total electric charge on the fluid sphere

$$q(r_0) = \varepsilon = \pm \frac{a_1 \gamma \alpha^{1/2} r_0^3}{\pi^{1/2} y_0^{3/2}}. \tag{49}$$

If we put $a = 0$ or $\gamma = 0$, the line-element goes over to the open cosmological model of Einstein-de Sitter, where space time is spatially flat and is infinite in extent; ε then vanishes.

The interior metric (1) can be matched across the moving boundary $r = r_0$ (in comoving co-ordinate) with the Reissner-Nordstrom solution

$$ds^2 = \left(1 - \frac{2m}{\bar{r}} + \frac{4\pi\varepsilon^2}{\bar{r}^2}\right) d\bar{t}^2 - \left(1 - \frac{2m}{\bar{r}} + \frac{4\pi\varepsilon^2}{\bar{r}^2}\right)^{-1} d\bar{r}^2 - \bar{r}^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad (50)$$

provided at the boundary (Faulkes (1969b))

$$p(r_0, t) = 0, \quad (51)$$

$$2m - \frac{4\pi\varepsilon^2}{r_0} e^{-\omega_0/2} = \tfrac{1}{4}\,\dot{\omega}_0^2 r_0^3 e^{3\omega_0/2 - \nu_0} - \tfrac{1}{4}\, r_0^2 \omega_0'^2 e^{\omega_0/2} - r_0^2 \omega_0' e^{\omega_0/2} \quad (52)$$

and Eqs (49) are satisfied. Here prime denotes derivative with respect to the radial co-ordinate. Further

$$\bar{r} = r_0 e^{\omega_0/2},$$

$$\bar{t} = \int e^{\nu_0/2}\left(1 + \frac{r_0\omega_0'}{2}\right)\left\{1 - \frac{2m}{r_0 e^{\omega_0/2}} + \frac{4\pi\varepsilon^2}{r_0^2 e^{\omega_0/2}}\right\}^{-1} dt. \quad (53)$$

From Eq. (52) we obtain

$$\dot{T}^2 = \frac{T^5}{S_0^3}\left[\frac{2m}{r_0^3} + \frac{4a_1\gamma\beta}{y_0^{3/2}}\right] - \frac{4a_1^2\gamma T^6}{S_0^3 y_0^3} \geqslant 0. \quad (54)$$

Since $\dot{R} = 0$ when $\dot{T} = 0$, the equality sign in (54) holds for the maximum or minimum volume of the fluid sphere. Hence for turning points we have from Eqs (45) and (51)

$$\ddot{T} = -\frac{2\gamma a_1^3 T^5}{S_0^3 y_0^3} \quad (55)$$

which reduces to

$$\ddot{S}_0 = -\frac{4\gamma a_1^3 T^5}{S_0^3 y_0^{7/2}} (S_0 + \alpha)^{1/2}. \quad (56)$$

Now, $\ddot{S}_0$ will have positive or negative values depending on the signs of the arbitrary constants a_1 and γ. Let us choose our arbitrary function T of t in such a way that $T > 0$. Since $e^{\omega/2} = S/T$ by (41a), we have $S > 0$. Further $e^{\nu/2} = Z/S$ by (42). Hence $Z > 0$. Further we must have $y > 0$ in order that ϱ and σ may be real. From Eqs (46) and (47) we find that ϱ is always greater than σ.

In order to find the condition that the volume of the 3-space is an extremum we must find out the turning points of $e^{\omega/2}$. The volume will be minimum or maximum depending on whether

$$\ddot{S}_0 T - S_0 \ddot{T} = \frac{2\gamma a_1^2 T^5}{S_0^2 y_0^3}\left[1 - \frac{2a_1 T}{S_0 y_0^{1/2}}(S_0 + \alpha)^{1/2}\right]$$

$$= -\frac{2\gamma a_1^2 T^5}{S_0^2 y_0^3} \cdot \frac{Z_0}{S_0} = -\frac{2\gamma a_1^2 T^5}{S_0^2 y_0^3} e^{\nu_0/2}$$

is positive or negative.

Case I. $\gamma > 0$; $a_1 > 0$. Here the turning point is a maximum volume and there will be a collapse to a singularity. From Eq. (46) the mass density is always positive throughout the distribution.

Again, differentiating Eq. (45) with respect to r, we obtain

$$8\pi p' = -\frac{8\gamma^2 a_1^2 T^4 r}{S^3 y^4 Z^2}\left[(\alpha-\beta^2)(a_1^2 T^2/y+2a_1 T\beta y^{-1/2})+3(\alpha-\beta^2)^2+2za_1 T\beta y^{-1/2}\right]$$

$$-(\dot{T}^2/T^2-\ddot{T}/T)\frac{8\gamma r}{Z^2}(2\alpha a_1^2 T^2 y^{-2}+S\beta a_1 T y^{-1/2}).$$

At the turning point $\dot{T} = 0$ and $\ddot{T}$ has the value given by Eq. (55). Under these conditions we find that p' is negative when $\beta > 0$ and $\alpha \geqslant \beta^2$. Hence pressure increases monotonically inward from zero value on the boundary. So pressure is everywhere positive inside the sphere when it has extremum volume. If we further assume that the arbitrary function of time T is such that T is negative, then the positivity of pressure is ensured everywhere at all instants of time.

Case II. $\gamma > 0$; $a_1 < 0$. Here also the turning point is a maximum volume and there will be a collapse to a singularity. The mass density remains always positive everywhere. But nothing definite can be said of the nature of pressure in this case. But if we further assume that $\beta < 0$ and $\alpha \geqslant \beta^2$ and $\ddot{T} < 0$, then the pressure always remains positive everywhere as in case I.

Case III. $\gamma < 0$, $a_1 > 0$. Here the turning point is a minimum volume and hence there is a bounce. If $\beta > 0$ and $\alpha \geqslant \beta^2$ the pressure remains positive everywhere at the turning point. However nothing definite can be said about the sign of pressure at other instants of time. Although the mass density becomes negative at the turning point one cannot be definite about the sign of density at other instants of time.

Case IV. $\gamma < 0$, $a_1 < 0$. The turning point being a minimum volume, it is again a case of bounce. If $\alpha \geqslant \beta^2$ and $\beta < 0$, then the nature of pressure and density is similar to the case III.

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