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MASS SPLITTING IN HIGH DIMENSIONAL EXTENSIONS OF GENERAL RELATIVITY

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A discussion of the relevance of high dimensional extensions of general relativity for the mass splitting problem in particle physics is given.

1. Introduction

During the past decades many attempts have been made to combine space-time ("external") and internal symmetries in a nontrivial way. "Nontrivial" in this context means (a) the total symmetry group is *not* a direct product of the external (Poincaré-) group and the internal group; (b) the total symmetry group allows for a mass splitting within internal symmetry multiplets as it is experimentally observed.

Powerful theorems [1–5] severely restrict the possibilities to solve these problems. It is possible, however, to solve (a) (but not (b)) as long as the total symmetry group is generated by a Lie algebra or a graded Lie algebra (of finite order) [2–6]. To be more precise: as long as the total symmetry group is generated by a (graded) Lie algebra which contains the Poincaré algebra as a subalgebra, the spectrum of the relativistically invariant mass operator is continuous or consists out of one point [3–6].

In this note I shall consider a Lagrangian describing "gravity" in a high dimensional superspace with commuting variables coupled to a scalar field. I shall demonstrate that this theory is a very hopeful candidate to solve the mass splitting problem ((b)).

2. The superspace

I shall consider a Riemannian superspace of $(N+4)$ dimensions. The variables will be labelled

$$(x_\mu, i_n), \quad n = 1, 2, \ldots, N, \tag{1}$$

and they all commute with each other.

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The *flat* structure of the space is as follows: the x_μ are elements of ordinary Minkowski space with metric say, diag $(-+++)$. The i_n I propose to consider as elements of the parameter space on which the elements of some given internal group G act. To proceed further it is to be observed that the "relative normalization" of coordinates is not yet fixed. By "relative normalization of coordinates" I mean the following: In special relativity the proper coordinates to make up an invariant line element are not $\vec{x}$ and t but $\vec{x}$ and ct and the line element in the metric given is

$$ds^2 = d\vec{x}^2 - c^2 dt^2. \tag{2}$$

Similarly I now write the line elements for the superspace

$$ds^2 = d\vec{x}^2 - c^2 dt^2 + \sigma b^2 \sum_{n=1}^{N} di_n^2. \tag{3}$$

In case the internal symmetry group is not simple but of the form say, $G = G_1 \times G_2$, one would replace this formula by

$$ds^2 = d\vec{x}^2 + c^2 dt^2 + \sigma_1 b_1^2 \sum_{n=1}^{N_1} di_n^2 + \sigma_2 b_2^2 \sum_{n=N_1+1}^{N} di_n^2. \tag{4}$$

$\sigma, \sigma_1, \sigma_2$ are sign factors.

It is now an important observation that the algebra of symmetry transformations which appears in the formulation of the so-called "no-go theorems" [1–6], is a (graded) Lie algebra (of finite order). All these (graded) Lie algebras have the particular property that their elements are *linear* transformations of the parameter space

$$\{(z_a) = (x_\mu, i_n), a = 0, 1, 2, \ldots, N+3\}. \tag{5}$$

There are solutions for the first problem among those sets of linear transformations [7–10]. There will be, however, no solutions for the second problem, (b) [3, 6].

Looking at the problem from this point of view it appears to be most natural to consider a theory which has a symmetry group the group of all (linear and nonlinear) transformations. General relativity is such a theory for the fourdimensional external space. Therefore I shall now consider an analogous theory in $(4+N)$ dimensions.

3. *Action and field equations*

I shall now consider the theory described by the following action

$$S = \int d^{4+N}z\{\sqrt{|g|}\,(R + L_\Phi)\} \tag{6}$$

with

$$L_\Phi = -\tfrac{1}{2}\,\Phi_{,a}\Phi^{,a} - \tfrac{1}{2}\,m^2\Phi^2. \tag{7}$$

All quantities here have the usual meaning except that they are to be understood in $(4+N)$ dimensions. The field equations following from (6) are

$$G_{ab} = R_{ab} - \alpha g_{ab} R = \kappa T_{ab} \tag{8}$$

and

$$\Phi_{,a}{}^{;a} - m^2 \Phi = 0. \tag{9}$$

Semicolon means covariant derivative. Consider now the last equation. In flat space it is the ordinary Klein-Gordon equation in $(N+4)$ dimensions and the spectrum of the Laplacian will be continuous. Equation (9) may be rewritten as

$$(\triangle - m^2)\Phi(z) = 0, \tag{10}$$

Δ being the Laplacian in $(N+4)$-dimensional curved space. It is a natural procedure to determine the mass spectrum from the eigenvalues of the Laplacian. Equation (10) then states the hypothesis that up to a universal constant the eigenvalues of the Laplacian are to be identified with masses.

An example will make clear how things will work. Let the superspace be $(N+4)$ dimensional flat space with metric diag $(- + + + \sigma \ldots \sigma)$ and $G =$ SO(N). The latter group acts on the N internal coordinates i_n. Then the above equation may be wirtten as

$$\left(\frac{\partial^2}{\partial x_\mu \partial x^\mu} + \frac{\sigma}{b^2} \sum_{n=1}^{N} \frac{\partial^2}{\partial i_n^2} - m^2\right) \Phi(x, i) = 0. \tag{11}$$

Thus we get a mass spectrum for the physical mass given by the eigenvalues of

$$\frac{\partial^2}{\partial x_\mu \partial x^\mu} \tag{12}$$

but it is continuous. But now the parameter space the elements of SO(N) are acting on may as well be the surface of the unit sphere in N dimensions (this being a curved $(N-1)$ dimensional space). Then we have

$$(\triangle_{\text{ext}} - \triangle_{\text{int}} - m^2)\Phi(x, i) = 0, \tag{13}$$

where Δ_{ext} is given by the expression (7) and Δ_{int} has the eigenvalues $L(L+N-2)$. $L = 0$, 1, 2, ... $\Phi(x, i)$ is given as the product of a plane wave in x and a spherical harmonic in i. The interpretation of L is fixed by the observation that Δ_{int} is the quadratic Casimir operator for SO(N).

High dimensional versions of general relativity are by no means a new object; they rather found continuous interest from the early days of general relativity on [11–24]. In particular the sample given above has been discussed in more detail by Rayski [13–16] and Cremmer et al. [20–22]. The latter authors show that an exact solution exists for the system (8), (9) with the properties described in the example, whereas Rayski discusses in particular the consequences for the classification problem in particle physics.

In the example given above the (constant) curvature has been inserted "by hand". One would like to determine the curvature by a set of dynamical equations. But precisely these equations are given by the (coupled) system (8) and (9). Whereas the masses generated by the mechanism described in the example are believed to be very large [13–16, 19–22], it has been speculated already by Rayski [15] that a nontrivial Riemannian structure (i. e., nonconstant curvature) may be related to the mass splitting in internal symmetry particle multiplets. The analysis of this note (including the discussion of the no-go theorems) increases this hope.

It would be beyond the purpose of this note to construct explicit examples for internal spaces with nontrivial Riemannian structure; it is however, elucidating to consider at least qualitatively such an example.

Consider a fourdimensional version of the field equation (9). It may be written as [25]

$$\frac{1}{\sqrt{-g}}\,\partial^{\mu}(\sqrt{-g}\;g_{\mu\nu}\partial^{\nu})\Phi - m^2\Phi = 0. \tag{14}$$

The Newtonian approximation is described by

$$g_{00} = 1 + \frac{2V}{c^2}, \tag{15}$$

$$g_{ik} = -\left(1 - \frac{2V}{c^2}\right)\delta_{ik}, \qquad i, k = 1, 2, 3, \tag{16}$$

$$V = -\frac{\kappa M}{r}. \tag{17}$$

Seeking solutions of (14) where the time dependence of Φ is given by a factor exp (iEt) one finds that for large $r^2 = \vec{x}^2$ equation (14) takes the form of the (nonrelativistic) Schrödinger equation for the H-atom,

$$\Delta_3\Psi + [k - U(r)]\Psi = 0, \tag{18}$$

$$U = -\frac{\alpha}{r}, \tag{19}$$

where Δ_3 is the threedimensional Laplacian, provided, $\partial\Phi/\partial r$ falls off rapidly enough for increasing r. A similar phenomenon has been observed [26] for the (exact) Schwarzschild solution.

In the Schrödinger problem bound states occur for $k < 0$, the spectrum of negative k's being discrete. In the present situation, however, k is to be replaced by E^2 which is positive, provided the flat metric is Minkowskian. If it is Euclidean, k is to be replaced by $-E^2$ being negative and we are concerned asymptotically with the bound state problem of the H-atom. An Euclidean solution may be useful if it is taken to describe the internal part of the space. A detailed analysis of these systems will be dealed with in forthcoming work.

It is elucidating to compare this approach with the usual discussion of spectrum generating wave equations (compare Ref. [27] for a review). The notorious problem in these approaches is to find arguments to write down the spectrum generating terms; it is precisely this problem the present approach promises to solve.

4. *Remarks concerning interpretation*

A central role in general relativity is played by the equivalence principle. The statement is that the gravitational interaction can be replaced by curvature of the space. In the formalism the principle appears as the statement that a pointlike particle moves along geodesics. Such a notion however, is not adequate for an elementary particle. The equation of motion for such a particle is the field equation (9). The similarity lies in the fact that again "interaction" is replaced by "curvature".

Curved superspaces have already been discussed in the literature [11–24]. The main motivation was to find a unified picture for gauge models like electrodynamics and general relativity. More recently [20–24] dual models were the motivation for studying high dimensional spaces. A traditional problem in the discussion of superspaces is that of the additional (internal) variables. The problem does not occur for anticommuting internal variables [7, 8] since these variables can be eliminated from the Lagrangian. A classical escape for commuting variables (compare, e. g., Ref. [19]) is to assume periodicity with slow variation for the unobserved variables; correspondingly the theory contains particles with very high masses [13]. Another possibility would be to impose boundary conditions such that asymptotically the solutions of the field equations do not depend any more on the internal variables. Yet another possibility would be to consider transition matrix elements which are integrated with respect to the internal variables, like in S-matrix theory where all variables which are not observed in a certain scattering experiment are summed over.

This interpretation problem, however, appears in any superspace theory and is not specific for the particular theory given here. As a matter of fact in particular the mixing of external and internal variables is just what one wants to do.

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REFERENCES

[1] S. Coleman, J. Mandula, *Phys. Rev.* **159**, 1251 (1967).
[2] L. O'Raifeartaigh, *Phys. Rev.* **139**, B1052 (1965).
[3] L. O'Raifeartaigh, *Phys. Rev. Lett.* **14**, 575 (1965).
[4] L. O'Raifeartaigh, *Phys. Rev.* **161**, 1571 (1967).
[5] L. O'Raifeartaigh, *Phys. Rev.* **164**, 2000 (1967).
[6] W. Mecklenburg, *Lett. Nuovo Cimento* **13**, 683 (1975).
[7] J. Wess, B. Zumino, *Nucl. Phys.* **70B**, 39 (1974).
[8] A. Salam, J. Strathdee, *Phys. Rev.* **D14**, 2830 (1976).
[9] A. Salam, J. Strathdee, ICTP Trieste, Preprint IC/76/12.
[10] P. Fayet, S. Ferrara, *Phys. Rep.* **32C**, 249 (1977).
[11] Th. Kaluza, *Sitz. Preuss. Akad.* 966 (1921).
[12] O. Klein, *Z. Phys.* **37**, 895 (1926).
[13] J. Rayski, *Acta Phys. Pol.* **27**, 89 (1965).
[14] J. Rayski, *Acta Phys. Pol.* **27**, 947 (1965).
[15] J. Rayski, *Acta Phys. Pol.* **28**, 87 (1965).
[16] J. Rayski, *Lett. Nuovo Cimento* **18**, 422 (1977).

[17] A. Komar, *Phys. Rev. Lett.* **13**, 220 (1965).
[18] R. Kerner, *Ann. Inst. Henri Poincaré* **9**, 143 (1968).
[19] W. Thirring, *Acta Phys. Austr. Suppl.* **9**, 256 (1972).
[20] E. Cremmer, J. Scherk, *Nucl. Phys.* **B103**, 399 (1976).
[21] E. Cremmer, J. Scherk, *Nucl. Phys.* **B108**, 409 (1976).
[22] E. Cremmer, J. Scherk, *Nucl. Phys.* **B118**, 61 (1976).
[23] Z. Horvath, L. Palla, E. Cremmer, J. Scherk, *Nucl. Phys.* **B127**, 57 (1977).
[24] J. F. Luciani, preprint LPTENS 77/9.
[25] L. D. Landau, E. M. Lifschitz, *Lehrbuch der theoretischen Physik*, Band II, 5., berichtige Auflage, Akademie-Verlag, Berlin 1971. Compare for the following in particular Sec. 86 and Sec. 106.
[26] D. G. Boulware, *Phys. Rev.* **D11**, 1404 (1975).
[27] V. L. Ginzburg, V. I. Manko, *Sov. J. Part. Nucl.* **7**, 1 (1976).