FUNDAMENTAL RELATIVISTIC ROTATOR*

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Professor Jan Weyssenhoff was Myron Mathisson's sponsor and collaborator. He introduced a class of objects known in Cracow as "kręciołki Weyssenhoffa", "Weyssenhoff's rotating little beasts". The Author describes a particularly simple object from this class. The relativistic rotator described in the paper is such that its both Casimir invariants are parameters rather than constants of motion.

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1. Historical introduction

As you know from the talk by Professor Trautman, there was a special relationship between Mathisson and my teacher, Professor Weyssenhoff. Socially Professor Weyssenhoff was, as we would say it today, Mathisson's sponsor. Scientifically, however, he was his follower. In particular, Professor Weyssenhoff's life-long fascination with rotating bodies was undoubtedly a result of his collaboration with Mathisson. I remember very well, if it is possible at all to remember something very well after more than 50 years, that Professor Weyssenhoff's fascinations were not always well received. This could be seen from the expression "kręciołki Weyssenhoffa", "Weyssenhoff's rotating little beasts", which was used to describe the object of his studies. Serious people were supposed to do nuclear physics. But 50 years is a lot in terms of human experience. Today nuclear physics does not seem to be a particularly exciting subject, while "kręciołki Weyssenhoffa" are, as I will try to convince you, a way to study some difficult and not well understood problems in special and general theory of relativity.

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2. The rigid body of Hanson and Regge

I shall not describe here Professor Weyssenhoff's contributions since they are actually quite well known and have been frequently quoted in scientific literature [1]. Instead I will start my consideration with the notion of relativistic rigid body introduced by Hanson and Regge [2]. According to Hanson and Regge a relativistic rigid body is a tetrad associated with a position and moving in accordance with some relativistically invariant laws of motion. It is a simple matter to see that a tetrad is fixed by three null directions. Thus the rigid body of Hanson and Regge can be equivalently characterized as a dynamical system described by position and three null directions. This gives nine degrees of freedom and an enormous variety of *a priori* possible relativistically invariant actions. It means also that nothing particularly useful can be obtained from so general a scheme.

3. A rotator as the simplest clock

In Newtonian physics the simplest clock is a rotator which consists of two point masses connected by a rigid and massless rod of length l. A rotator is not a rigid body of Euler. This is seen from the fact that a rotator has five degrees of freedom while an Eulerian rigid body has six degrees of freedom. $5 = 3 + 2$. This admittedly simple observation leads me to the following

Definition 1. *A relativistic rotator is a dynamical system described by position and a single null direction and, additionally, by two parameters, m (mass) and l (length).*

I will denote by x the space-time radius vector and by k the null direction. I will also denote by ab the scalar product of vectors a and b:

$$ab = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3 \,.$$

It is a simple matter to see that the most general relativistically invariant action for a system described by position x and null direction k has the following form:

$$S = \int d\tau (-) m \sqrt{\dot{x}\dot{x}} f \left(l^2 \frac{\dot{k}\dot{k}}{(k\dot{x})^2} \right) . \tag{1}$$

Here τ is an arbitrary parameter, a dot denotes differentiation with respect to τ and f is an arbitrary function. It would be dangerous to choose this function at random since we know from the work of Professor Weyssenhoff that this can lead to pathologies such as superluminal motions. In the following I will fix the function f in a way suggested by the famous work of Wigner [3].

Wigner says that relativistic quantum mechanical systems should be classified by means of unitary, irreducible representations of the Poincaré group. Unitarity is a difficult notion which has no classical counterpart. Irreducibility, however, is a simple algebraic notion which does have its classical counterpart. In our context irreducibility means that both Casimir invariants of the Poincaré group should have fixed numerical values *i.e.* they should be parameters rather than constants of motion. This leads me to the following

Definition 2. *A dynamical system is said to be phenomenological if its Casimir invariants are constants of motion. A dynamical system is said to be fundamental if its Casimir invariants are parameters, not constants of motion.*

To apply this definition I perform the following calculation: for the action (1) I calculate the Noether constants of motion P_μ and $M_{\mu\nu}$, the Casimir invariants $P_\mu P^\mu$ and $W_\mu W^\mu$, where

$$W_\mu = -\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}M^{\nu\rho}P^\sigma$$

is the Pauli–Lubański pseudovector. I put forward the requirement that both Casimir invariants should be parameters *i.e.* should not depend on initial conditions. This gives me two differential equations for one function of one real variable $\xi = l^2(\dot{k}\dot{k})/(k\dot{x})^2$. Remarkably enough, both equations can be simultaneously solved and this gives the result summarized in the following

Theorem. *There is only one relativistic rotator which is fundamental. Its Hamilton's action has the form*

$$S = \int d\tau(-)m\sqrt{\dot{x}\dot{x}}\sqrt{1+\sqrt{-l^2\frac{\dot{k}\dot{k}}{(k\dot{x})^2}}}\,. \tag{2}$$

For this rotator

$$P_\mu P^\mu = m^2\,, \qquad W_\mu W^\mu = -\frac{1}{4}m^4 l^2\,.$$

4. Some applications

Eq. (2) is a beautiful formula. The beauty is connected in part with its property of being uniquely determined by a set of well defined ideas. Beautiful formulae are important for theoretical physics because they convey in a compact form many ideas which might be difficult to elucidate separately.

But Eq. (2) can also be used to discuss some important theoretical issues, for example, the so called clock hypothesis, which says that a moving clock shows its proper time even when accelerated. Professor Weyssenhoff did not believe in the clock hypothesis. While explaining this to us he would take off his watch and say: look, this clock now does indeed show its proper time, however, if I drop it, it surely will not show its proper time anymore. This common sense approach was probably a trace of the fact that Professor Weyssenhoff started his scientific career as an experimental physicist. He did his Ph.D. Thesis at ETH in Zurich. Einstein quotes this Thesis in his little known monograph on Brownian motions. Professor Weyssenhoff and Einstein knew each other very well, so we might guess that Einstein's own position on the clock hypothesis was similar.

One has to realize that all sufficiently accurate real clocks are very complex devices. Moreover, they work the way they do because they are quantum mechanical objects. By contrast, Eq. (2) describes a simple classical device which is an ideal clock. It is ideal in two different ways in which this word can be understood: it is ideal because it is perfect, experiences no fatigue or friction, and it is ideal because it is a mathematical construct not related to a real mechanism. To check the clock hypothesis for the ideal device described by Eq. (2) one has to perform the following calculation: one has to assume that a certain combination of $x(\tau)$ and $k(\tau)$ is given because the clock is accelerated by some external force, and to calculate the motion of the "pointer" $k(\tau)$ on the unit sphere of null directions. This calculation is actually quite difficult but it is clear that to obtain anything resembling the clock hypothesis one must assume that the motion of the entire device is adiabatic with respect to the motion of the "pointer" $k(\tau)$. "Adiabatic" means that the null direction k must perform an extremely large number of cycles during each time in which the velocity of the entire device changes in a significant way. I am sure that this condition is necessary but I do not know if it is sufficient.

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