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# ON THE RELATIONS BETWEEN FIELD EQUATIONS AND EQUATIONS OF MOTION IN NEWTONIAN AND EINSTEINIAN GRAVITY*

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A pseudo-field theoretic reformulation of the Newton–Euler dynamics of isolated, gravitating fluids is given. The basic equations of that theory are shown to be regular limits of Einstein's gravitational field equation. It is reviewed how the equations of motion for mass points can be obtained as approximations from those for extended bodies without use of a regularisation to remove infinities. Finally Einstein's (1916–1918) approximation method is revived; its similarity to Newtonian theory suggests the possibility of avoiding infinities also in General Relativity.

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## 1. Introduction

Two important related and characteristic aspects of general relativity (GR) theory are the nonlinearity of the field equation

$$G^{\alpha\beta} = 8\pi T^{\alpha\beta} \tag{1}$$

in the potentials $g_{\alpha\beta}$ and its first derivatives, and the divergence condition

$$T^{\alpha\beta}{}_{;\beta} = 0\,. \tag{2}$$

This is imposed by the field equation on those laws which specify the kind of matter (including non-gravitational fields) which is supposed to interact,

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via its energy-momentum-stress tensor $T^{\alpha\beta}$, with the gravitational field. For a fluid in isentropic motion, *e.g.*, where

$$T^{\alpha\beta} = (\mu + p)U^\alpha U^\beta + pg^{\alpha\beta}\,, \qquad \mu = \rho u,\ \ p = \rho^2 \frac{du}{d\rho}\,, \tag{3}$$

Eq. (2) implies the GR-analogues of Euler's equations of fluid motion for energy density $\mu$ and 4-velocity $U^\alpha$, the pressure $p$ being given implicitly as a function of $\mu$. In this and similar cases there is no room for equations of motion independent of the field equation; Cauchy data for the fields $g_{\alpha\beta}$, $\mu$, $U^\alpha$ uniquely determine how the fields evolve.

There is another significant fact about Eq. (1): In GR, bodies of matter have to be modeled as spatially extended objects; material points are incompatible with (1) as shown already by Schwarzschild's vacuum solution.

In celestial mechanics one is primarily interested in motions of bodies as a whole. An appropriate description therefore requires representing each body by a point respectively world line contained in the (convex hull of the) support of the energy, or mass density of the body, and equations determining these world lines.

For solving, or even formulating, problems of both kinds of equations of motion in GR — those referring to hydrodynamics such as stellar collapse or accretion disks, and those concerning motions of separated bodies as, *e.g.*, components of binaries or gyroscopes — the close similarity between GR and Newtonian mechanics is used for guidance. Problems of the second type include test body motions the treatment of which owes essential ideas to Myron Mathisson; see [2] and the references given there.

The following sections are intended to throw some light on the subjects mentioned in the introduction. They present the topics announced in the abstract.

## 2. A pseudo-field theoretic formulation of Newton–Euler gravitational dynamics

Traditionally the theory of an isolated, gravitating ideal fluid has been based on the following laws:

$$\partial_t \rho + \partial_a(\rho v^a) \;=\; 0\,, \tag{4}$$

$$\rho(\partial_t v^a + v^b \partial_b v^a + \partial^a U) + \partial^a p \;=\; 0\,, \tag{5}$$

$$\mathit{\Delta} U \;=\; 4\pi\rho\,, \tag{6}$$

$$\text{if } |\boldsymbol{x}| \to \infty,\ \ U(\boldsymbol{x}) \to 0 \ \text{ and } \ \rho(\boldsymbol{x}) \to 0\,. \tag{7}$$

For definiteness and in view of relativity I take the fluid to be compressible, and I restrict my attention to isentropic motions, so that the pressure $p$ depends on the mass density $\rho$ according to the equation of state

$$p = \rho^2 \frac{du}{d\rho}\,, \tag{5a}$$

in which $u(\rho)$ denotes the energy/mass ratio at a fixed value of the specific entropy. Then $\rho, v^a, U$ are the theory's independent fields.

In consequence of (4) and (6), the Euler equation (5) may be replaced by the local conservation law for momentum,

$$\partial_t(\rho v^a) + \partial_b\Big(\rho v^a v^b + p\delta^{ab} + \frac{1}{4\pi}\left[\partial^a U \partial^b U - \tfrac{1}{2}\partial_c U \partial^c U \delta^{ab}\right]\Big) = 0\,, \tag{8}$$

in which appears not only matter pressure, but the gravitational pressure tensor, too.

This "mechanical" theory contains local conservation laws for mass and momentum; a corresponding energy law does not give additional information. The twofold role of $\rho$ as "attracting" and "attracted" mass is expressed in (6) and (5), respectively. More important is the fact that we neither know non-stationary solutions of (4)–(7) nor proofs of existence theorems; the difficulty is due to the mixed elliptic-hyperbolic character of the PDE system (4)–(6).

In the traditional formulation, the derivative of the potential $U$ generated by the density via (6) and (7), occurs in the equation of motion (5). I shall now introduce an alternative formulation in which the equations of motion follow from "field equations". For this purpose I introduce vector and tensor potentials through

$$\Delta W^a = 4\pi\rho v^a\,, \qquad \Delta Z^{ab} = 4\pi S^{ab}\,,$$

where $S^{ab}$ stands for the total momentum flux density of Eq. (8), which contains matter and field contributions; the potentials are supposed to fall off like $U$ in (7). (The fall-off conditions should be specified by choosing suitable function spaces such that $\Delta^{-1}$ exists and provides unique potentials for sources. This can be done only in connection with existence and uniqueness proofs. Here I am assuming this to be possible.)

Next, consider the constraint equations

$$\Delta U = 4\pi\rho\,, \tag{9}$$

$$\Delta W^a = 4\pi\rho v^a\,, \tag{10}$$

$$\Delta Z^{ab} - \partial^a U \partial^b U + \tfrac{1}{2}\delta^{ab}\partial_c U \partial^c U = 4\pi(\rho v^a v^b + p\partial^{ab})\,, \tag{11}$$

and the evolution equations

$$\partial_t U + \partial_a W^a = 0\,, \tag{12}$$

$$\partial_t W^a + \partial_b Z^{ab} = 0\,. \tag{13}$$

Note that in (9) to (11) the right hand sides contain matter variables only, while (12) and (13) are linear and contain only potentials.

Assuming appropriate fall-off conditions, it is easily verified that the fields $\{U, W^a, Z^{ab}, \rho, v^a\}$ satisfy (9)–(13) if and only if $\{\rho, v^a, U\}$ satisfy (4)–(6) and $W^a, Z^{ab}$ are defined as above.

In this formulation the "field equations" (9)–(13) imply the equations of motion; the crucial, nonlinear equation providing gravitational interaction via field stresses is (11). I call this a "pseudo"-field formulation since the potentials do not propagate but are determined instantaneously by the matter distribution.

If the field equations are linearized in the potentials, the interaction is suppressed, and matter moves under the influence of its pressure only. If, however, the linear "mass constraint" (6) ($\equiv$ (9)) is solved to provide $U[\rho]$ and the result is put into the Euler equation (5), gravitational interaction is taken into account.

## 3. Derivation of the field formulation of Newton–Euler dynamics from general relativity

For several purposes it is useful in GR, to employ the variables $\boldsymbol{g}^{\alpha\beta} = \sqrt{-g}g^{\alpha\beta}$, and to impose the harmonic gauge condition

$$\partial_\beta \boldsymbol{g}^{\alpha\beta} = 0\,. \tag{14}$$

((14) can be chosen locally without loss of generality. The argument which follows assumes global validity.)

Since the present aim is to get from GR to Newtonian theory, I assume the existence of one-parameter families of GR spacetimes close to Minkowski spacetime for which, in inertial coordinates $x^a, t$

$$\widehat{\boldsymbol{g}}^{\alpha\beta} = \mathrm{diag}(c, c, c, -c^{-1})\,.$$

With $\lambda = c^{-2}$ as parameter, I then introduce for GR-spacetimes of such a family, the "potentials" $U^{\alpha\beta}(x^a, t, \lambda)$ via

$$\sqrt{\lambda}\,(\boldsymbol{g}^{\alpha\beta} - \widehat{\boldsymbol{g}}^{\alpha\beta}) =: 4\lambda^2 U^{\alpha\beta}\,. \tag{15}$$

With these notations, the harmonically reduced Einstein field equation takes the form

$$\Big(\Box + 4\lambda^2 U^{\gamma\delta}\partial_\gamma\partial_\delta\Big)U^{\alpha\beta} + Q^{\alpha\beta}\Big[U^{\alpha\beta}, (\partial U^{\alpha\beta})^2, \lambda\Big] = 4\pi T^{\alpha\beta}\Big(1 + \lambda P[U^{\alpha\beta}, \lambda]\Big)\,, \tag{16}$$

and (14) reads

$$\partial_\beta U^{\alpha\beta} = 0\,. \tag{17}$$

((16) is obtained by rewriting the $\boldsymbol{g}^{\alpha\beta}$-version of Einstein's equation, used by Fock [4], Landau–Lifshitz [7] and others, in the notation introduced above.)

In (16), $\Box$ is the flat-space wave operator, $\Box = \Delta - \lambda^2 \partial_{tt}^2$; the $Q^{\alpha\beta}$ are quadratic in $\partial_\gamma U^{\alpha\beta}$ and polynomial in $\lambda$, of degree 0 in $\lambda$ for spatial indices, and otherwise of positive degree. $P$ depends on the potentials and is polynomial of degree 0 in $\lambda$.

Equation (16) is a semilinear wave equation in the unknowns $U^{\alpha\beta}$ for $\lambda > 0$ (GR). The equation remains well defined for $\lambda = 0$, and simplifies in this case, with the appropriate $T^{\alpha\beta}$, to the "Newtonian" constraint equations (9)–(11), with the identification

$$U^{00} = U\,, \qquad U^{0a} = W^a\,, \qquad U^{ab} = Z^{ab}\,.$$

For $\lambda = 0$ the harmonic condition (17) is identical to the "Newtonian" evolution equations (12), (13). Thus, if a sequence of GR-solutions can be parametrized as assumed here, and if its potentials $U^{\alpha\beta}$ converge for $\lambda \to 0$ then the limit functions will represent a Newtonian solution. The deep question is which Newtonian solutions can be so represented or, equivalently, whether one can use the theorem to establish the existence of GR-solutions for some $\lambda > 0$, given a Newtonian one. In fact, Uwe Heilig [6] proved the existence of GR-solutions representing fluid bodies in rigid, stationary rotations, using (16), (17).

## 4. The center-of-mass motion of well-separated, extended bodies in Newtonian dynamics

Let there be a finite number of fluid (or elastic) bodies separated by empty space, and let $B$ denote the (time dependent) domain of space occupied by one of the bodies, a connected component of the support of $\rho$ at any time $t$, assumed compact. The mass center of that body is then $X^a$,

$$MX^a = \int_B \rho x^a dV\,. \tag{18}$$

For comparison with GR I shall use in the following the field formulation of Newtonian theory given in Section 2.

Applying $\partial_t$ to (9), $\partial_a$ to (10), and using (12) gives the local mass conservation law (4). That law asserts:

*The mass measure $dm = \rho dV$ is invariant with respect to the fluid flow; it is a measure on the 3-manifold of fluid elements. In particular,*

$$\int_B dm = M \tag{19}$$

*is constant.*

Eq. (19) implies

$$M\dot{X}^a = \int_B \rho v^a dV \,. \tag{20}$$

Next, applying $\partial_t$ to (10), $\partial_t$ to (11), and using (13) gives the local momentum conservation law (8). At this stage, all "field equations" have been used.

Now, integrating local momentum conservation over $B$ supplies

$$\frac{d}{dt}P^a = \frac{d}{dt}\int_B \rho v^a dV = -\int_B \rho \partial^a U dV \,,$$

where again (9) has been used. With (20), the last equation can be rewritten as

$$M\ddot{X}^a = -\int_B \partial^a U dm \,. \tag{21}$$

Using the linearity of Poisson's equation we decompose the potential $U$ into the self potential $U_\mathrm{s}$ of the body and the rest, the potential $U_\mathrm{e}$ external to the body to be considered, taking the appropriate densities in (6). Then

$$M\ddot{X}^a = -\int_B (\partial^a U_\mathrm{s} + \partial^a U_\mathrm{e}) dm \,. \tag{22}$$

The self force vanishes due to *actio* = *reactio*,

$$\int_B \rho \partial^a U_\mathrm{s} dV = \int_{B\times B'} \frac{\rho(x)\rho(x')}{|x-x'|^3}\,(x^a - x'^a) dV dV' = 0 \,.$$

Therefore,

$$M\ddot{X}^a = -\int_B \rho \partial^a U_\mathrm{e} dV \,. \tag{23}$$

$U_\mathrm{e}$ is analytic in a neighbourhood of $B$, hence

$$\ddot{X}^a = -(\partial^a U_\mathrm{e})(X^o)\left(1 + O\Big(\Big(\frac{d}{D}\Big)^2\Big)\right), \tag{24}$$

if the body has linear dimension $d$, the other bodies of the system have distances of order $D$, and $d \ll D$. In the step from (23) to (24), the vanishing of the dipole term, which results from the center-of-mass definition, has been used.

In (24), the quadrupole and higher multipole terms can be explicitly displayed, giving the force per unit mass as an absolutely convergent series.

I have reviewed these well-known facts in a slightly new form to recall, how in Newtonian theory approximate equations of motion for the mass centers follow without the introduction of fictitious mass points and the resulting infinities in the potential. Essential are the definition of a mass center, the splitting of the potential, and the vanishing of the self forces.

The center of mass achieves two things. It not only provides a point representing the body, but it also removes the dipole part from the force which ensures a small error in (24).

A body may be idealized as a test body if *(i)* its mass is negligible compared to the masses of other relevant bodies, and *(ii)* if the body is much smaller than its distances to the heavy bodies; then the error term in (24) can be neglected, and (24) expresses the universality of free fall; it describes then, in fact, a geodesic of the Newtonian connection.

## 5. Linearized Einstein gravity and equations of motion

It is usual to call a gravitational field in GR "weak", if, in suitable coordinates, the metric takes the form

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} - \tfrac{1}{2}\eta_{\alpha\beta} h \,, \tag{25}$$

where $h = \eta^{\alpha\beta} h_{\alpha\beta}$ and $|h_{\alpha\beta}| \ll 1$, and to linearize Einstein's field equation w.r.t. the variables $h_{\alpha\beta}$, which results in

$$\Box h^{\alpha\beta} - 2\partial^{(\alpha} h^{\beta)} + \eta^{\alpha\beta}\partial_\gamma h^\gamma = -16\pi T^{\alpha\beta} \,. \tag{26}$$

Flat spacetime is taken as a background for the fields $h^{\alpha\beta}$, which are considered as small of first order, and indices are shifted with the Minkowski metric.

Eq. (26) implies

$$T^{\alpha\beta}{}_{,\beta} = 0 \,, \tag{27}$$

the linearized version of (2).

In consequence of the general covariance of the exact field equation, the l.h.s. of (26) is invariant under linear gauge transformations, so that one may put, without loss of generality,

$$h^\alpha := \partial_\beta h^{\alpha\beta} = 0 \,. \tag{28}$$

Then (26) simplifies to the "reduced" linearized equation

$$\Box h^{\alpha\beta} = -16\pi T^{\alpha\beta} \,. \tag{29}$$

In contrast to (26), (29) is hyperbolic and can be solved, whether or not $T^{\alpha\beta}$ obeys (27); but (29) has to be complemented by (28) if the linearized equation (26) is to be satisfied exactly.

Equation (27) implies that matter — as far as the energy-momentum distribution is concerned — is not affected by the (weak) gravitational field. This conclusion corresponds to the analogous one in Newtonian theory. More generally, one recognizes: If the field equations of a field theory imply equations of motion for its sources, the field will mediate interactions between sources only if the field equation is nonlinear. Because of this fact, it is somewhat inappropriate to call the formalism based on (26) and (27), or (28) and (29), a linear "theory of gravity".

To overcome the "somewhat troublesome" fact signified by Eq. (27), one has to preserve some nonlinearity of the underlying full theory, if only to obtain gravitational interactions for slowly moving, weakly stressed, widely separated objects like the bodies of the solar system. Actually this has been achieved already be Einstein in his gravitational wave papers of 1916 and 1918 which, surprisingly, are rarely fully exploited; Pauli in his famous article of 1921 [8] being an exception.

The way out is based on two ideas: *(i)* Do not solve the linearized equation exactly, rather try to solve the exact equation approximately. *(ii)* Use, even at the lowest level of approximation, a nonlinear approximation to (2) along with (29) and (28).

To follow *(ii)*, rewrite the exact equation (2) in the form

$$\partial_\beta(\sqrt{-g}\,T^\beta_\alpha) = \frac{1}{2}(\partial_\alpha g_{\beta\gamma})\sqrt{-g}\,T^{\beta\gamma}\,, \tag{30}$$

used by Einstein ever since 1913. This form resembles Euler's equation, with $(\partial_a U)\rho$ on the r.h.s., and it suggests introducing into the r.h.s. of (30) the weak potential from (25). This leads to

$$\partial_\beta([1-\tfrac{1}{2}h]T^{\alpha\beta}) = -\frac{1}{32\pi}(\Box h^{\beta\gamma})\partial^\alpha(h_{\beta\gamma}-\tfrac{1}{2}\eta_\alpha h)\,, \tag{31}$$

to be solved in second order. Remarkably, the r.h.s. can be converted into a divergence; (31) is equivalent to the (approximate) local conservation law

$$\partial_\beta\left((1-\tfrac{1}{2}h)T^{\alpha\beta}+t^{\alpha\beta}\right)=0\,, \tag{32}$$

where, by definition,

$$t_{\alpha\beta}=(32\pi)^{-1}\left(-\partial_\alpha h_{\lambda\mu}\partial_\beta h^{\lambda\mu}+\tfrac{1}{2}\partial_\alpha h\partial_\beta h+\tfrac{1}{2}\eta_{\alpha\beta}\left[\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu}-\tfrac{1}{2}\partial_\lambda h\partial^\lambda h\right]\right). \tag{33}$$

Eqs. (29) and (32) imply that $\Box h^\alpha$ is of second order, and thus $h^\alpha$ is of second order, too. If therefore $h_{\alpha\beta}$ and matter variables obey (29) in first, and (32) in second order, the field equation (26) is satisfied to first order, too. We should thus consider, rather than (26), the pair (29), (31) as an approximate theory of gravitation. Its linear part, (29), serves to express the field in terms of matter variables, and its nonlinear part determines the evolution, in close analogy to Newtonian theory. The role played in Newtonian theory by local momentum conservation is taken in GR by the approximate local energy-momentum conservation. The similarity of the two theories could be made even closer by including in the former an energy conservation law, in the latter a conservation of baryonic mass, which would provide an invariant mass measure.

In view of the similarity considered here it would seem to be possible to avoid the introduction of fictitious mass points in GR, too. This may not lead to practical improvements, but to a better understanding of the interplay between matter and field in GR.

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