HAMILTONIAN DESCRIPTION OF MOTION OF CHARGED PARTICLES WITH SPIN*

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Equations of motion of a classical charged particle carrying a non-vanishing internal angular momentum (spin) are derived from first principles in the special relativistic context. The equations are implied by the conservation law of both the energy-momentum four-vector and the angular-momentum tensor carried by the total physical system, composed of the particle and the field. Our method leads directly to the variational and the hamiltonian formulations of the dynamics. It is based on the programme formulated in Kijowski, *Gen. Relativ. Gravitation J.* **26**, 167 (1994) and *Acta Phys. Pol. A* **85**, 771 (1994) and may be treated as an implementation of the idea of "deriving equations of motion from field equations", formulated by Einstein.

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1. Introduction

When thinking about the problem of motion of a particle with spin, people often accept the following, naive point of view: 1. External forces (*e.g.* electromagnetic) act on the particle's trajectory. 2. Once the trajectory is "decided" by the above action, the spin propagates "as much parallelly as possible" along the trajectory, *i.e.* undergoes the Fermi propagation. Of course, the parallel propagation is excluded for any non-trivial trajectory because the "spin vector" s^λ, representing the internal angular momentum of the particle, remains orthogonal with respect to the four-velocity u^λ of the particle: $u^\lambda s_\lambda \equiv 0$.

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As will be seen in the sequel, such a naive "law of dynamics" contradicts the angular momentum conservation. To restore this conservation, a non-vanishing contribution p_μ to the particle's kinetic momentum P_μ, due to the presence of the spin, must be taken into account. This contribution contains the particle's acceleration a, namely:

$$p_\mu = \varepsilon_{\lambda\kappa\nu\mu} u^\lambda a^\kappa s^\nu \, . \tag{1}$$

Consequently, the conservation of the total kinetic momentum "$P_\mu + p_\mu$" implies an extra force acting on the trajectory of the particle and resulting from the time derivative of p_μ. Due to the structure of (1), this force contains the third derivative $\dddot{x}$ of the particle position, contained in the quantity $\dot{a}$. The equations of motion contain, therefore, a "three dots" term. Contrary to the better known "three dots force", proposed by Dirac (see [2]), the force due to the spin is *orthogonal* to the four-velocity of the particle and does not produce any self-acceleration (*cf.* the so called "runaway solutions" of the Dirac equation). We call this force the Mathisson[1] force (see [3]). The goal of this paper is to derive this force in the purely special-relativistic context and to show that the resulting dynamics is conservative. Our procedure uses a natural variational principle. As a final result we obtain the canonical (symplectic) structure of the phase space of the "particle with spin" system and we calculate the Hamiltonian function generating its dynamics. In fact, we first test the applicability of our method on the case of a particle without spin and derive the Lorentz force acting on it. This way we show that the Lorentz force cannot be treated as an independent law of physics, but is a necessary consequence of Maxwell equations, describing the electromagnetic field interacting with "particle-like" sources (*cf.* [5]). Finally, we show how to generalise this method to the case of a particle with spin.

There is an apparent paradox when we want to formulate a "three-dots-equation" in a Hamiltonian way. The independent variables of such a theory seems to be: three positions, three velocities and three accelerations (second order derivatives of the position), together with the two configurations of the spin vector $\vec{s} \in \mathbb{R}$; $\|\vec{s}\| = s$, (a three-vector with a fixed length s). Hence, we have $3+3+3+2 = 11$ parameters — an uneven number which apparently contradict the possibility of a Hamiltonian description, where the dimensionality of the total phase space (number of configurations + number of corresponding momenta) is always even. The answer to this paradox consists in the observation that the Mathisson force is always orthogonal to the spin. Therefore, the motion in the direction of the spin remains

[1] A similar force was earlier discussed by Frenkel (*cf.* [4]) in the context of the idea of a "rotating electron". According to this idea, the classical rotation of an extended portion of electric charge produces a non-vanishing magnetic moment. In Frenkel's approach, however, equations of motion *are not derived* but postulated *a priori*.

governed by the "double-dot" law, rather then the "three-dots". When formulated in a variational way, the dynamics exhibits a one-parameter group of gauge transformations due to the fact that the "three-dots" enter into the Lagrangian function of the system only *via* the vector product $\dot{\vec{a}} \times \vec{s}$. Consequently, the Legendre transformation leads to a degenerate canonical structure on the above 11-dimensional space: one of these parameters is a pure gauge. The physical phase space of the system is 10-dimensional and can be obtained as a quotient space with respect to this degeneracy.

A similar, but much simpler, construction for a particle without spin was proposed in [6]. It can be noticed that the idea of the construction which we propose in the present paper was already contained there, but the technical problems related to the explicit description of this quotient space proved to be too difficult at that time. Only few month ago I was able to quotient the non-physical, 11-dimensional phase space with respect to the gauge transformations and to calculate explicitly both the physical phase space and the Hamiltonian function of the system. This way we have at our disposal a powerful method of Hamiltonian analysis which, hopefully, will allow us to verify and, maybe, to strengthen the results and the hypothesis proposed by Mathisson (*cf.* [3]).

2. Relativistic field theory in an arbitrary non-inertial reference frame

The present section contains a simple exercise in relativistic field theory. The techniques which we develop here enable us to construct a system of coordinates which proves to be extremely useful for our purposes. As will be seen in the sequel, using this tool we are able to solve explicitly all the constraints of the theory and to describe the remaining "true degrees of freedom" in a relatively simple way.

Consider a Lagrangian field theory in the Minkowski spacetime M (*e.g.* electrodynamics interacting with some matter fields). The dynamics of the theory is encoded within a special-relativistic Lagrangian density $L = L(\psi, \partial\psi)$, where by ψ we denote symbolically all the fields appearing in the theory.

Let $y^k = q^k(t)$ with $k = 1, 2, 3$; $t = y^0$, be the coordinate description of a given time-like world line ζ with respect to a laboratory frame, *i.e.* to a system (y^μ), $\mu = 0, 1, 2, 3$; of Lorentzian space-time coordinates. The line ζ represents a trajectory of a non-inertial observer.

We will construct an accelerated reference frame, co-moving with the observer ζ. For this purpose let us consider at each point $(t, q^k(t)) \in \zeta$ the 3-dimensional hyperplane Σ_t orthogonal to ζ, *i.e.* orthogonal to the four-velocity vector $U(t) = (u^\mu(t))$:

$$(u^\mu) = (u^0, u^k) := \frac{1}{\sqrt{1-\boldsymbol{v}^2}}\,(1, v^k)\,, \tag{2}$$

where $v^k := \dot{q}^k$, $v^0 = 1$, $(v^\mu) = (1, \boldsymbol{v})$ and $\boldsymbol{v}^2 = \|\boldsymbol{v}\|^2$ is the Euclidean length of a three vector $(v^k) = \boldsymbol{v}$. We call Σ_t the "rest frame surface". Choose on Σ_t any system (x^i) of *cartesian* coordinates, centered at the particle's position (*i.e.* the particle is located at the point $x^i = 0$).

We consider space-time as a disjoint sum of rest frame surfaces Σ_t, each of them corresponding to a specific value of the coordinate x^0 and parameterised by the coordinates (x^i). For points belonging to Σ_t we put $x^0 := t$. This way we obtain a new system $(x^\alpha) = (x^0, x^k)$ of "co-moving" coordinates in a neighbourhood of ζ. Our construction implies that both coordinates x^0 and y^0 coincide on the trajectory ζ. In general, (x^α) is not a global system because different Σ's may intersect. Nevertheless, we will use it *globally* to describe the evolution of the field ψ from one Σ_t to another. For a hyperbolic field theory, initial data on one Σ_t implies the entire field evolution. We are allowed, therefore, to describe this evolution as a one-parameter family of field initial data over subsequent Σ's.

Formally, we will proceed as follows. We consider an abstract space-time $\mathbb{M} := T \times \Sigma$ defined as the product of an abstract time axis $T = \mathbb{R}^1$ with an abstract, three dimensional Euclidean space $\Sigma = \mathbb{R}^3$. Given a world-line ζ, we will need an identification of points of $\mathbb{M}$ with points of the physical space-time M. Such an identification is not unique because on each Σ_t we are still free to choose an arbitrary O(3)-rotation.

Suppose, therefore, that an identification F has been chosen, which is *local* with respect to the observer's trajectory. By locality we mean that, given the position and the velocity of the observer at time t, the isometry

$$F_{(q^k(t), v^k(t))} : \Sigma \mapsto \Sigma_t \tag{3}$$

is uniquely defined, which maps $0 \in \Sigma$ into the particle position $(t, q^k(t)) \in \Sigma_t$.

As an example of such an isometry which is *local* with respect to the trajectory we can take the following one. Choose the unique boost transformation $\mathcal{B}$ relating the laboratory time axis $\partial/\partial y^0$ with the observer's proper time axis U. Next, define the observer's space axis $\partial/\partial x^k$ on Σ_t by acting with $\mathcal{B}$ on the corresponding laboratory space axis $\partial/\partial y^k$. It is easy to check (*cf.* [6]), that the resulting formula for $F_{(q^k(t), v^k(t))}$ reads:

$$y^0(t, x^l) \;:=\; t + \frac{1}{\sqrt{1-\boldsymbol{v}^2(t)}}\; x^l v_l(t)\;, \tag{4}$$

$$y^k(t, x^l) \;:=\; q^k(t) + \left(\delta^k_l + \varphi(\boldsymbol{v}^2) v^k v_l\right) x^l\,. \tag{5}$$

Here, the following function of a real variable has been used:

$$\varphi(\tau) := \frac{1}{\tau}\left(\frac{1}{\sqrt{1-\tau}}\ -1\right) = \frac{1}{\sqrt{1-\tau}(1+\sqrt{1-\tau})}\,. \tag{6}$$

The function is regular (even analytic) for $\tau = \boldsymbol{v}^2 < 1$. The operator $(\delta^k_l + \varphi(\boldsymbol{v}^2)v^k v_l)$ acting on rest-frame variables x^l comes from the boost transformation.

Suppose, therefore, that for a given trajectory ζ a *local* isometry (3) has been chosen, which defines $F_\zeta : \mathbb{M} \mapsto M$. This mapping is usually not invertible: different points of $\mathbb{M}$ may correspond to the same point of space-time M because different Σ_t's may intersect. It enables us, however, to define the metric tensor on $\mathbb{M}$ as the pull-back $F^*_\zeta g$ of the Minkowski metric. The components $g_{\alpha\beta}$ of the above metric are defined by the derivatives of F_ζ, *i.e.* they depend upon the first and the second derivatives of the position $q^k(t)$ of our observer.

Because (x^k) are cartesian coordinates on Σ, the space–space components of g are trivial: $g_{ij} = \delta_{ij}$. The only non-trivial components of g are, therefore, the lapse function and the (purely rotational) shift vector:

$$N \ = \ \frac{1}{\sqrt{-g^{00}}} = \sqrt{1-\boldsymbol{v}^2}\ (1 + a_i x^i)\,, \tag{7}$$

$$N_m \ = \ g_{0m} = \sqrt{1-\boldsymbol{v}^2}\ \epsilon_{mkl}\omega^k x^l\,, \tag{8}$$

where a^i is the observer's acceleration vector in the co-moving frame. The rotation ω^m depends upon the coordination of isometries (3) between different Σ_t's. Because ω^m depends locally upon the trajectory, it may also be calculated in terms of the velocity and the acceleration of the observer, once the identification (3) has been chosen. In the case of example (4)–(5), it is easy to check that

$$a^i \ = \ \frac{1}{1-\boldsymbol{v}^2}\left(\delta^i_k + \varphi(\boldsymbol{v}^2)v^i v_k\right)\dot{v}^k\,, \tag{9}$$

$$\omega_m \ = \ \frac{1}{\sqrt{1-\boldsymbol{v}^2}}\,\varphi(\boldsymbol{v}^2)v^k\dot{v}^l\epsilon_{klm}\,, \tag{10}$$

where $\dot{v}^k$ is the observer's acceleration in the laboratory frame.

The metric $F^*_\zeta g$ is degenerate at singular points of the identification map, where the identification is locally non-invertible because adjacent Σ's intersect (*i.e.* where $N = 0$), but this degeneration does not produce any difficulty in what follows.

The simplest O(3)-coordination of the isometries (3) would be: $\omega^m \equiv 0$. It implies the Fermi-propagating of the coordinates x^k along ζ. Such a coordination is, however, non-local with respect to the trajectory. Indeed, the identification F_t between Σ and Σ_t would be, in this case, a result of the Fermi propagation of a given mapping F_{t_0} from the initial time t_0 to the actual time t. Such a mapping cannot be described by a local formula (3). We stress, however, that for our construction we do not need to specify any coordination F, provided it is *local*.

Using the metric (7)–(8) on $\mathbb{M}$, we may rewrite the invariant Lagrangian density L of the field theory under consideration, just as in any other curvilinear system of coordinates. The Lagrangian obtained this way depends upon the field ψ, its first derivatives, but also on the observer's position, velocity and acceleration, which enter *via* the metric components. Variation with respect to ψ produces field equations in the co-moving coordinate system (x^α). Due to the relativistic invariance of the theory, its action, equal to the integral of the above quantity, does not change for any variation (with fixed boundary!) of the observer's trajectory. This implies that the variation of L with respect to the observer's position q^k is trivial and *does not* produce new field equations.

For our purposes we will keep, however, at the same footing the field degrees of freedom ψ and (at the moment, physically irrelevant) observer's degrees of freedom q^k:

$$L = L(\psi, \partial\psi; q, \dot{q}, \ddot{q}) .$$

For such a Lagrangian theory, we perform the Legendre transformation in the field variables, and pass to the Hamiltonian description of the dynamics, keeping description of the "mechanical" degrees of freedom on the Lagrangian level. For this purpose we define

$$L_{\mathrm{H}} := L - \pi\dot{\psi} , \tag{11}$$

where $\dot{\psi}$ denotes derivative with respect to x^0 and π is the momentum canonically conjugate to ψ:

$$\pi := \frac{\partial L}{\partial \dot{\psi}} . \tag{12}$$

The quantity $L_{\mathrm{H}} = L_{\mathrm{H}}(\psi, \pi; q, \dot{q}, \ddot{q})$ plays role of a Hamiltonian function (with negative sign) with respect to the field configurations and momenta (ψ, π), whereas it remains the Lagrangian density for the observer's position q^k. It is an analog of the *Routhian function* in analytical mechanics. It generates the hamiltonian field evolution with respect to the accelerated frame, if the "mechanical degrees of freedom" q^k are fixed. Due to (7)–(8), this evolution is a superposition of the following three transformations:

- time-translation in the direction of the local time-axis of the observer,
- boost in the direction of the acceleration a^k of the observer,
- purely spatial O(3)-rotation ω^m.

It is, therefore, obvious that the numerical value of the generator $L_{\rm H}$ of such an evolution is equal to

$$L_{\rm H} = -\sqrt{1-\boldsymbol{v}^2}\left(\mathcal{H} + a^k\mathcal{R}_k - \omega^m\mathcal{S}_m\right) , \tag{13}$$

where $\mathcal{H}$ is the rest-frame field energy (generator of the time translations), $\mathcal{R}_k$ is the rest-frame static moment (generator of the boosts) and $\mathcal{S}_m$ is the rest-frame angular momentum (generator of the rotations), all of them calculated at the point $(t, q^k(t))$. The two vectors $\mathcal{R}_k$ and $\mathcal{S}_k$ represent the angular momentum tensor $\mathcal{M}^{\alpha\beta}$ according to formulae: $\mathcal{M}^{k0} = \mathcal{R}^k$ and $\mathcal{M}^{ij} = \epsilon^{ijk}\mathcal{S}_k$. The factor $\sqrt{1-\boldsymbol{v}^2}$ in front of the generator is necessary, because the time $t = x^0 = y^0$, which we use to parameterise the observer's trajectory, is not the proper time along ζ but the laboratory time.

Now, we want to convince the reader that, similarly as was true in case of L, Euler–Lagrange equations obtained when varying $L_{\rm H}$ with respect to the observer's position $q(t)$ are satisfied *identically* if the field equations are satisfied. The proof follows directly from the conservation laws of the total four-momentum $\mathcal{P}^\alpha$ and the total angular momentum $\mathcal{M}^{\alpha\beta}$ of the field, implied by Noether's theorem. Indeed, the four-momentum conservation reads:

$$\nabla_0\mathcal{P}^\alpha = \dot{\mathcal{P}}^\alpha + \Gamma^\alpha_{0\beta}\mathcal{P}^\beta = 0 \, , \tag{14}$$

where $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols of the metric $g_{\alpha\beta}$, calculated on the trajectory, *i.e.* at $x^k = 0$. Putting $\mathcal{P}^0 = \mathcal{H}$ and calculating Γ's from (8), one immediately obtains the following "accelerated-frame version" of Noether conservation laws:

$$\dot{\mathcal{H}} = -\sqrt{1-\boldsymbol{v}^2}\, a^k\mathcal{P}_k \, , \tag{15}$$

$$\dot{\mathcal{P}}_k = \sqrt{1-\boldsymbol{v}^2}\left(-a_k\mathcal{H} - \epsilon_k{}^{ml}\omega_m\mathcal{P}_l\right) . \tag{16}$$

The angular momentum conservation reads $\nabla_0\widetilde{\mathcal{M}}^{\alpha\beta} = 0$, where:

$$\widetilde{\mathcal{M}}^{\alpha\beta} := \mathcal{M}^{\alpha\beta} - y^\alpha\mathcal{P}^\beta + y^\beta\mathcal{P}^\alpha \ ,$$

because the quantity $\mathcal{M}^{\alpha\beta}$ describes the angular momentum with respect to the *moving* point $(t, q^k(t))$; to obtain a conserved quantity we must recalculate it with respect to any *fixed* point, *i.e.* $(0, 0)$, and the result of this recalculation is the above tensor $\widetilde{\mathcal{M}}$. This way we obtain the following form of the conservation laws:

$$\dot{\mathcal{R}}_k = \sqrt{1-\boldsymbol{v}^2}\left(\mathcal{P}_k - \epsilon_{kim}a^i\mathcal{S}^m - \epsilon_{kil}\omega^i\mathcal{R}^l\right) , \tag{17}$$

$$\dot{\mathcal{S}}^m = \sqrt{1-\boldsymbol{v}^2}\left(\epsilon^{mil}a_i\mathcal{R}_l - \epsilon^{mij}\omega_i\mathcal{S}_j\right) . \tag{18}$$

These conservation laws are implied by field equations. It has been proved in [6], that the Euler–Lagrange equations obtained by varying (13) with respect to the observer's position $q(t)$ are equivalent to equations (15)–(18).

The fact, that the Euler–Lagrange equations of the theory are not independent, is typical for a gauge theory. This property may be nicely described in the Hamiltonian picture. Considering L_{H} as the generator of the evolution of both the field degrees of freedom and the observer's degrees of freedom, we may perform the Legendre transformation also with respect to the latter, and find this way the complete Hamiltonian of the entire (observer + field) system (see again [6] for details). It may be proved, that q plays the role of a gauge parameter: momenta canonically conjugate to observer's position are not independent but subject to constraints. Reducing the theory with respect to these constraints we would end up with the "true" degrees of freedom, namely those describing the field. Fixing the observer's trajectory would play role of "gauge fixing" and the "evolution equations" of the observer would be *automatically satisfied* if the field equations were satisfied.

3. Particle as a field "soliton"

We assume that our field theory describes matter field (fields?), ϕ interacting with the electromagnetic field $f_{\mu\nu}$. Symbolically, we write: $\psi = (\phi, f)$. Moreover, we assume that for weak electromagnetic field and vanishing matter fields the theory coincides with Maxwell electrodynamics. Finally, we suppose that the theory admits a static, stable ("soliton-like") solution, for which matter fields are concentrated within a tiny region nearby to a point (the "strong-field-region"), whereas outside of it (in the "weak-field-region") the matter fields practically vanish and the electromagnetic field is sufficiently weak to be described by the Maxwell equations. By stability of the solution we mean that its field configuration corresponds to a local minimum of the total energy, treated as a functional on the phase space of Cauchy data of the theory. This implies that the amount of energy carried by small perturbations of this solutions does not differ considerably from the total energy (mass) of the unperturbed soliton, which we denote by m. We interpret this solution as the charged particle at rest, surrounded by its own Coulomb field. We call the parameter m the rest mass of the particle. Observe, that m is not a local quantity! It contains not only the energy of the "material core" of the particle, but also electromagnetic contribution carried by its Coulomb tail $\left(\vec{D} = \frac{e\vec{x}}{\|x\|^3}; \vec{B} = 0\right)$ and, finally, the interaction energy. Hence, m is "already renormalised" and there is no sense to split it into the sum "$m = m_0 + m_{\mathrm{C}}$", where m_0 would be the "bare mass" which, finally, "gets dressed by a cloud of fotons" and m_{C} would be the mass of this cloud. Contrary to the mass, the total electric charge e of the particle is a local quantity, contained completely in the strong field region. Moreover, the soliton may carry some amount of the angular momentum s^k. It is also a local quantity, because in the weak field region the field reduces to the Coulomb tail, which does not carry any angular momentum. We call s^k the *spin of the particle*. Of course, the total momentum p^k carried by the static solution vanishes.

Acting with the Poincaré group on the soliton solution we obtain, due to relativistic invariance of the theory, a 6-parameter family of boosted and spatially shifted solitons, describing particles moving with constant velocity, *i.e.* free particles.

We are interested, however, in the problem of motion of particles moving with an arbitrary velocity — not necessary constant. As a model of such a particle we take any solution of our field theory which fulfils the following conditions: 1. there is a time-like trajectory ζ and a tiny world tube T around it, containing the entire "strong field region" of the solution; 2. on each rest frame surface Σ_t, field configuration within T does not differ considerably from the field configuration of the "free particle" *i.e.* from the (boosted) soliton solution. The above conditions mean that the complement $\complement T = M \setminus T$ of T is the weak field region. Hence, outside of T the field dynamics reduces to Maxwell equations.

We are going to show in the sequel that not every trajectory ζ admits such a solution. Its existence (*i.e.* a possibility, that the particle can move along ζ) implies a certain condition on the trajectory. We are going to derive these "equations of motion" from field equations. This means that we use in our derivation Maxwell equations as the *unique quantitative* assumption (plus several *qualitative* assumptions, some of them being already formulated above).

Consider, therefore, such a "moving particle solution". As shown in Sec. 2, it fulfils the variational principle defined by the "Lagrango-Hamiltonian" L_{H} given by formula (13). Being free to choose the observer's trajectory, we simply choose it equal to the particle's trajectory ζ. We shall calculate now the value of L_{H}. On each rest frame surface Σ_t the Cauchy data (ϕ, π) for the matter field vanish outside of the strong field region $T \cap \Sigma_t$ and does not differ considerably from the data (ϕ_0, π_0) corresponding to the soliton solution. On the other hand, the electromagnetic field on Σ_t may be decomposed into the sum: $f = f_0 + \tilde{f}$, where f_0 is the field corresponding to the soliton solution and $\tilde{f} := f - f_0$ is simply the remainder. The quantities $\mathcal{H}$, $\mathcal{P}$, $\mathcal{R}$ and $\mathcal{S}$ are obtained *via* integration of appropriate components of the total energy-momentum tensor of the theory. Therefore, they may be decomposed as a sum: integration over the strong field region $T \cap \Sigma_t$ gives the contributions which we call $\mathcal{H}_T$, $\mathcal{P}_T$, $\mathcal{R}_T$ and $\mathcal{S}_T$ and the integration over its complement gives, respectively, the contributions which we call $\mathcal{H}_{\complement T}$, $\mathcal{H}_{\complement T}$, $\mathcal{R}_{\complement T}$ and $\mathcal{S}_{\complement T}$. But, in the weak field region the only significant component of the theory is the electromagnetic field f and the total energy-momentum tensor reduces to the Maxwell tensor which is bilinear with respect to the field f. Hence, we have a further decomposition:

$$\mathcal{H}_{\complement T} = h_{\complement T} + H + \mathbb{H}\,,$$

where the first term is quadratic in f_0, the second is bilinear and the last one is quadratic in $\tilde{f}$, and similarly for the quantities $\mathcal{P}$, $\mathcal{R}$ and $\mathcal{S}$. Summing up the terms quadratic in f_0 with the contributions from the strong field region (*e.g.* $h = \mathcal{H}_T + h_{\complement T}$ for the energy), we obtain the following decomposition:

$$\begin{array}{rcl} \mathcal{H} & = & h + H + \mathbb{H}\,, \\ \mathcal{P}_k & = & p_k + P_k + \mathbb{P}_k\,, \\ \mathcal{R}_k & = & r_k + R_k + \mathbb{R}_k\,, \\ \mathcal{S}_k & = & s_k + S_k + \mathbb{S}_k\,. \end{array} \tag{19}$$

These quantities contain the entire complicated, internal structure of the extended moving particle. We want to construct a simplified theory which — disregarding most of this structure — is able to capture in a cumulative way the dynamics of the particle.

For this purpose we are going to approximate the above exact quantities by something which is much simpler. Because the contributions from the strong field region do not differ considerably from the corresponding integrals for the soliton, the first column may be well approximated by the values describing the particle at rest: $h = m$, $p_k = 0$ and $r_k = 0$ (the last equation because we choose the center of mass of the soliton as the origin of our coordinate system)[2].

Also the remaining two columns may be calculated approximatively without any knowledge about the entire internal structure of the particle. Indeed,

[2] Actually, the origin of the system could be put either at the center of mass or at the center of charge. Here, we use a simplifying assumption that both the centers coincide. Admitting a non vanishing distance between them could be a good remedy against the non-stability of the model, see [7]. On the other hand, a careful analysis of the renormalisation procedure used here shows that the distance between the two centers must be treated as a dynamical quantity, see [8].

we have approximately $f = f_0$ inside the strong field region. But, outside of $T \cap \Sigma_t$, we know that f_0 is the Coulomb field. Let us replace it by the Coulomb field f_{C} on the entire Σ_t. Consider the Maxwell field $f_{\mathrm{M}} := f_{\mathrm{C}} + \tilde{f}$, with delta-like source concentrated on ζ. Its Maxwell energy-momentum tensor has a non-integrable singularity on ζ and the straightforward integration cannot be used to define its energy-momentum and the angular momentum. However, it *can be used* to calculate an approximative value of the second and the third column of (19), if we integrate only terms bilinear in f_{C} and $\tilde{f}$ (for the second column) and quadratic in $\tilde{f}$ (for the third column). Moreover, it is easy to show that H and S_k calculated this way vanish identically (*cf.* [1]).

This way we are lead to the theory of Maxwell field f_{M} with δ-like sources, *i.e.* theory of *point particles*, treated as an approximation of the exact theory of *extended particles*. The "energy-momentum and angular-momentum content" of the total "field + particles" system is described by the following free of singularities ("already renormalised") quantities:

$$\begin{aligned} \mathcal{H} &= m + 0 + \mathbb{H}\,, \\ \mathcal{P}_k &= 0 + P_k + \mathbb{P}_k\,, \\ \mathcal{R}_k &= 0 + R_k + \mathbb{R}_k\,, \\ \mathcal{S}_k &= s_k + 0 + \mathbb{S}_k\,. \end{aligned} \tag{20}$$

In the first column we have the "pure particle" quantities, characterising the free particle[3], *i.e.* the soliton solution of the exact solution of the non-linear theory (ϕ, f). And this is the only, phenomenological input of the non-linear theory. To calculate the second and the last column we split (on each Σ_t separately) the linear Maxwell field f_{M} into the Coulomb tail of the particle: $f_{\mathrm{C}} = \left(\vec{D} = \frac{e\vec{x}}{\|x\|^3}; \vec{B} = 0\right)$, and the remaining field $\tilde{f} = f_{\mathrm{M}} - f_{\mathrm{C}}$. Next, we calculate Maxwell energy-momentum tensor and skip all the terms which are quadratic in the Coulomb tail f_{C}, because the particle's own field has already been taken into account in the first column. The last column is defined by integrals of terms quadratic with respect to $\tilde{f}$. It describes the "field quantities". The second column is defined by integrals of mixed terms and is responsible for interaction between particle and field. The arguments presented above show that (20) provides a good approximation of the exact quantities (19). Conservation laws (15)–(18) for the total quantities are equivalent to the variational principle for the trajectory, with the Lagrangian

[3] As discussed in the previous footnote, a non-vanishing, dynamical static moment $r^k = r^k(\vec{a})$, describing the distance between center of charge and center of mass could also be taken into account. It turns out that the consistency of the theory implies a certain dependence of the rest mass upon the acceleration. These terms — negligible for small accelerations — may stabilise considerably the dynamics of the system (*cf.* [7]). These issues will be discussed elsewhere.

given by formula (13). Together with Maxwell equations for f_{M}, with the delta-like current $j^\mu := 4\pi e u^\mu \delta_\zeta$, they define a consistent theory of interacting particles and fields, which was proposed in [1]. It was proved in [9] that the initial value problem is well posed in this theory.

4. Test particle model

In the present paper I will restrict myself to a further approximation of the above complete theory. This approximation may be called a "test particle" theory. It is obtained by assuming that the "external field" $\tilde{f}$ is a free Maxwell field given *a priori.* This means that we neglect the radiation field and, besides of the external field, we take into account only the particle's own Coulomb field. Of course, such a truncation of the field is a flagrant violation of the Maxwell equations. Such a violation is, nevertheless, much less severe than the procedures based on the truncation of the particle's Coulomb tail, which is usually done in the test particle theory. In our approach, the use of the variational principle based on the Lagrangian (13) implies equations of motion which guarantee that this violation will be "as slight as possible". Of course, the applicability of such an approximation is restricted to the situation when the acceleration of the particle remains sufficiently small during the evolution. To describe also the radiation due to the motion of the particle, the previous version of the theory must by taken into account (see also discussion in [10]).

In the test particle model, the last column of (20) represents the energy-momentum and the angular-momentum of the free Maxwell field, namely the external field $\tilde{f}$. These quantities *are already conserved*, due to Maxwell equations. Therefore, as noticed in Sec. 2, the contribution to Euler–Lagrange equations due to these terms is trivial. Hence, we may skip them when calculating the Lagrangian (13). We end up with the following Lagrangian describing the dynamics of test particles in a given field $\tilde{f}$:

$$L_{\mathrm{H}} = -\sqrt{1-\boldsymbol{v}^2}\left(m + a^k R_k - \omega^m s_m\right) . \tag{21}$$

The case without spin (*i.e.* when $s = 0$) was already analysed in paper [5]. The following identity was proved:

$$\begin{aligned} L_{\mathrm{H}} &= -\sqrt{1-\boldsymbol{v}^2}\left(m + a^k R_k\right) \\ &= -m\sqrt{1-\boldsymbol{v}^2} + e v^\mu A_\mu + \text{boundary terms}\,, \end{aligned} \tag{22}$$

where A_μ denote potentials for the external field $\tilde{f}$. Of course, the left hand side does not depend upon the choice of potentials (it is gauge invariant!) and so must be the right hand side. This means that the boundary terms must compensate the gauge dependence of the interaction term $e v^\mu A_\mu$.

The boundary terms may be skipped. This way we have proved the equivalence of the *gauge invariant but second order* (because containing acceleration) variational principle L_{H} and the standard, *gauge dependent but first order* variational principle:

$$L_{\text{gauge}} = -m\sqrt{1-\boldsymbol{v}^2} + ev^{\mu}A_{\mu}\,, \tag{23}$$

producing the Lorentz equations of motion for test particles. Hence, we have derived equations of motion from field equations for particles without spin! This result convinces us that an analogous procedure may be applied in the non-vanishing spin case.

Observe that the relation between the two equivalent Lagrangians is similar to the relation between the *gauge invariant but second order* Hilbert Lagrangian and the *gauge dependent but first order* Einstein Lagrangian in general relativity theory. They differ by boundary terms only.

5. Heuristic derivation of the Mathisson force

Before we discuss the Hamiltonian structure of the dynamics implied by the complete Lagrangian (21) for a particle with spin, we want to give a heuristic explanation of the origin of the Mathisson force. Our discussion is valid not only for the test particle model but also for a complete theory of interacting particles and fields.

Consider first a theory without spin. It was proved in [1] that equations of motion are equivalent to the energy-momentum conservation laws (15)–(16), whereas the angular momentum conservation (17)–(18) is obtained freely from them, provided Maxwell equations are fulfilled by the external field $\tilde{f}$. Now, we want to introduce also a non-vanishing spin s. This means that we must replace the quantity $\mathcal{S}$ in (18) by the sum $\mathcal{S}+s$. The equation will remain satisfied if the terms which we add on its both sides are equal. Hence, the spin must propagate according to the following law:

$$\dot{s}^m = -\sqrt{1-\boldsymbol{v}^2}\epsilon^{mij}\omega_i s_j \tag{24}$$

which is nothing but the Thomas precession. This means that we have Fermi propagation of the spin vector s along the particle's trajectory. The same substitution $\mathcal{S} \to \mathcal{S}+s$ in equation (17) leads to the violation of the angular momentum conservation, unless the new term is compensated by an extra contribution

$$p_k := \epsilon_{kim}a^i s^m \tag{25}$$

to the total momentum, due to the spin. Admitting such a contribution, the angular momentum conservation law remains fulfilled. The substitution

$\mathcal{P} \to \mathcal{P} + p$ does not lead to the violation of the energy conservation (15) because p is orthogonal to the acceleration a. However, the same substitution in the momentum conservation (16) produces the three-dots force coming from the time derivative of a in (25).

6. Phase space for a particle with spin

Consider, therefore, the Lagrangian for a test particle with spin:

$$L = L(q, \dot{q}, \ddot{q}|s^m) = -m\sqrt{1-\boldsymbol{v}^2} + ev^\mu A_\mu(q) + \varphi(\boldsymbol{v}^2) v^k \dot{v}^l \epsilon_{klm} s^m \,, \qquad (26)$$

equivalent (up to boundary terms) to the gauge invariant quantity (21). Variation of the last term with respect to the particle's position produces the Mathisson force. As discussed in Sec. 2, L is a Routhian function: Lagrangian with respect to the particle variables $(q, \dot{q}, \ddot{q})$ and Hamiltonian with respect to the field. Here, the spin is carried by the field. We conclude that the canonical structure of the spin variables s^m must be obtained from the canonical structure of the Cauchy data for the field. Consider, therefore, the infinite dimensional phase space of these data. Take a single point in it, corresponding to our soliton (particle at rest) centered at zero. Acting with the rotation group SO(3), we generate a three dimensional subspace Π of our infinite dimensional phase space. This is the phase space for the spin. Its points may be labelled by the value of the configuration of the spin vector $(s^m) = \vec{s}$ on a two dimensional sphere and, possibly, by an internal angle α. (The latter variable is irrelevant *a priori* if the soliton exhibits an axial symmetry, but we do not need such an assumption for our purposes.) The symplectic form of Π comes from the restriction to Π of the canonical symplectic form on the space of the field Cauchy data. It must be degenerate because Π is odd-dimensional. We know, moreover, that the three components of the spin vector form the Lie algebra $o(3)$ and, therefore, satisfy the standard Poisson brackets:

$$\{s^m, s^n\} = \epsilon^{mnk} s_k \,.$$

These two facts imply uniquely the canonical structure of Π: its symplectic form is degenerate in the direction of the phase α and is proportional to the surface form on the sphere describing the spin configuration $\vec{s} = (s^k)$:

$$\Omega_{\text{spin}} = \int_\Sigma d\pi \wedge d\psi \Big|_\Pi = -\frac{1}{2s^2}\, s^i \epsilon_{ijk}\, ds^j \wedge ds^k = s\, d(\cos\theta) \wedge d\varphi \,, \qquad (27)$$

where (θ, φ) are angular coordinates on the sphere $\{\|\vec{s}\| = s\}$ (from now on, we switch to the "arrow notation" of three vectors). Due to this formula, the

Hamiltonian equations of motion for the spin imply its rotation around the following angular velocity vector:

$$\tilde{\omega}_m = \varphi(\boldsymbol{v}^2) v^k \dot{v}^l \epsilon_{klm}$$

because the last term of (26) is equal to $\tilde{\omega}_m s^m = \omega s \cos\theta$. Hence, we have $\dot{\varphi} = \omega$. This is precisely the Thomas precession. The phase α is a pure gauge parameter and must be skipped from our considerations.

7. Hamiltonian function for a particle with spin

To pass to the Hamiltonian formulation of the dynamics we perform the Legendre transformation in the orbital variables (q^k), leaving the canonical description of the spin variables unchanged. Such a transformation is not completely standard because L is of the second differential order. (A simple description of the canonical structure for theories with higher order Lagrangians may be found in [11].) The transformation consists in considering velocities v^k as additional variables, related with position q^k by the constraint equation $\dot{q}^k - v^k = 0$. Finally, the Hamiltonian function is obtained from the Lagrangian by the standard formula:

$$H = H(q, v, P, Q, s) = P_k \dot{q}^k + Q_k \dot{v}^k - L\,, \tag{28}$$

where P_k and Q_k are momenta canonically conjugate to q^k and v^k, respectively. In particular, we have:

$$Q_k := \frac{\partial L}{\partial \dot{v}^k} = \varphi(\boldsymbol{v}^2) s^l v^m \, \epsilon_{klm}\,. \tag{29}$$

The symplectic structure of the theory is given by:

$$\Omega = \, dP_k \wedge \, dq^k + \, dQ_k \wedge \, dv^k + \Omega_{\text{spin}}\,. \tag{30}$$

We have here $4\times 3+2 = 14$ parameters. However, we must take into account 3 constraint equations (29), which reduce the number of parameters to 11. The phase space splits, therefore, into the standard, 6 dimensional "orbital" phase space, parameterised by positions and momenta and equipped with the canonical form

$$\Omega_{\text{orbital}} = d\,P_k \wedge d\,q^k\,, \tag{31}$$

and the remaining 5 dimensional "internal" space $\mathbb{R}^3 \times S^2$, parameterised by variables $(\vec{v}, \vec{s})$; $\|\vec{s}\| = s$, and equipped with the form:

$$\Omega_{\text{internal}} = \, dQ_k \wedge \, dv^k + \Omega_{\text{spin}}\,, \tag{32}$$

where Q_k are not independent variables, but are given by (29). We have, therefore:

$$\Omega_{\rm internal} = d\left(\varphi(\boldsymbol{v}^2)s^l v^m \epsilon_{klm}\right) \wedge dv^k - \frac{1}{2s^2}\, s^i \epsilon_{ijk}\, ds^j \wedge ds^k \,. \tag{33}$$

Of course, $\Omega_{\rm internal}$ is degenerate (because the dimension of this space is odd). The physical phase space of the "true degrees of freedom" is obtained as the quotient space of $\mathbb{R}^3 \times S^2$ with respect to this degeneracy. For this purpose we calculate the kernel of (33), *i.e.* a non-vanishing vector field X fulfilling equation: $i_X \Omega_{\rm internal} = 0$. It may be checked by inspection that the following field X (or any other of the form fX, where $f \neq 0$) satisfies this condition:

$$\begin{aligned} X \;=\; & \left[v^k - s^k \frac{\vec{v}\cdot\vec{s}}{s^2}\right] \frac{\partial}{\partial s^k} \\ & + \frac{\sqrt{1-\boldsymbol{v}^2}}{s^2}\left[\left(1+\sqrt{1-\boldsymbol{v}^2}\right) s^i - (\vec{v}\cdot\vec{s})\, v^i\right] \frac{\partial}{\partial v^i} \,. \end{aligned} \tag{34}$$

Next step of this reduction consists in calculating integral curves of this field. Finally, we identify points which belong to the same integral curve.

Recently, I was able to perform this programme and to give an explicit description of the quotient space. It is based on the observation that in every equivalence class $[(\vec{v},\vec{s})]$ of the element $(\vec{v},\vec{s})$, there is a *unique* representative $(\vec{w}, s\cdot\vec{\sigma})$, $\|\vec{\sigma}\| = 1$; such that $\vec{w}$ is orthogonal to $\vec{\sigma}$, *i.e.* fulfilling equation $\vec{w}\cdot\vec{\sigma} = 0$. We may, therefore, describe the quotient by the four dimensional space of pairs of mutually orthogonal vectors $(\vec{w},\vec{\sigma})$, and such that $\vec{\sigma}$ is normalised. The symplectic form (33), when restricted to this space, reduces to:

$$\tfrac{1}{s}\ \Omega_{\rm internal} = \mathrm{d}\left(\varphi(w^2)\sigma^l w^m \epsilon_{klm}\right) \wedge \mathrm{d}w^k - \frac{1}{2}\sigma^i \epsilon_{ijk} \mathrm{d}\sigma^j \wedge \mathrm{d}\sigma^k \,. \tag{35}$$

I am also able to diagonalise this form, *i.e.* to write it in terms of two pairs of mutually conjugate variables (this result will be presented elsewhere). Finally, using the above variables, the value of the Hamiltonian function may be calculated:

$$H = \vec{\pi}\cdot\vec{w} + \sqrt{1-w^2}\cdot\sqrt{m^2 + (\vec{\pi}\cdot\vec{\sigma})^2} + U(q)\ , \tag{36}$$

where $U(q) = -eA_0(q)$ is the scalar potential and $\vec{\pi} = (\pi_k)$ is the "kinetic momentum", defined in terms of the "canonical momentum" P_k (*cf.* formula (31)) by the following, standard formula:

$$\pi_k := P_k - eA_k(q)\ .$$

It is interesting to notice, that one of the equations generated by this Hamiltonian reads:

$$\vec{v} = \dot{\vec{q}} = \frac{\partial H}{\partial \vec{P}} = \frac{\partial H}{\partial \vec{\pi}} = \vec{w} + \vec{\sigma} \cdot \text{const.}$$

It gives a nice consistency test for the correctness of the above reduction procedure. Indeed, it is easy to check that this vector belongs to the same class as w.

8. Concluding remarks

We stress that no naive idea of a "rotating electron" was necessary for our purposes. In our approach, the angular momentum carried by the static, soliton solution of field equation ("spin") does not need to be imagined as a mechanical quantity, generated by a "rotation", but rather as having a field-theoretical nature. In particular, its value is fixed by the specific, non-linear properties of the field theory — the building material of the particle. Contrary to the mechanical "spin", where we can always add an infinitesimal amount of rotation, the quantity s is quantised: any deformation of the soliton solution destroys its static character. It is not difficult to prove that already relatively simple (but highly non-linear) models, like *e.g.* charged scalar field interacting with electromagnetic field, admit such solutions. For this purpose one can easily observe, that for a given shape of a solution it is possible to find such a Hamiltonian function, for which the shape in question is a local minimum of the total energy. Of course, not every Hamiltonian function leads to a relativistic theory, when the Legengre transformation back to the variational formulation is performed. But a careful analysis of these issues shows, that there is still enough room to fulfil also the relativity requirement.

Our approach, however, permits to disregard this (possibly very complicated) internal structure and to treat the three numbers characterising the particle: m, e and s, as phenomenological quantities. The equations of motion obtained this way are universal.

The autonomous Hamiltonian system described by (31) and (35)–(36) was derived in the "test particle limit". To obtain the consistent theory of interacting particles and fields one cannot treat the "external field" $\tilde{f}$ as a free Maxwell field given *a priori* but, rather, as additional degrees of freedom of the total system, *cf.* [1] and [6].

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