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EQUATIONS OF MOTION IN THE GAUGE GRAVITY MODELS*

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In the gauge gravitational models, the geometry of a spacetime manifold becomes non-Riemannian. The curvature, torsion and nonmetricity are all nontrivial in these models, in general. The study of the dynamics of the physical matter (particles, bodies, continuous media, *etc.*) in such manifolds is crucial for determining the actual geometrical structure of the spacetime. Here we briefly describe a model of a test particle with hypermomentum which can be used as a tool for detecting the non-Riemannian geometry, and recall that the conservation laws in the gauge gravity theories underlie the general analysis of the equations of motion in such models.

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1. Introduction

Dynamics of particles (and more generally, of bodies and continuous media) is determined by their physical properties and by the coupling to the external fields. The specific feature of the gravitational theories, that distinguishes them from the other field-theoretic models, is that the equations of motion should not be postulated separately but are derived from the field equations and from the conservation laws. This is well known for Einstein's general relativity which is based on the Riemannian geometry, and the same is true for the gauge gravity models which are formulated on the non-Riemannian spacetimes.

The importance of the study of the equations of motion is explained by their role as the tools for exploring the spacetime geometry. Einstein [1]

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stressed the experimental nature of the spacetime structure: "…The question whether this continuum has a Euclidean, Riemannian, or any other structure is a question of physics proper which must be answered by experience, and not a question of a convention to be chosen on grounds of mere expediency."

In the gauge approach to gravity (for the introduction and the overview see [2,3], for example) the gravitational field arises as a field of a Yang–Mills type. Analogously to the usual Yang–Mills theory which is based on the local symmetry group that acts in the internal space, the gauge gravity is also based on a local symmetry group which, however, acts directly on the spacetime manifold. Accordingly, whereas the standard Yang–Mills theory is naturally interpreted in terms of the nontrivial geometry of fiber bundles over a flat manifold, the gauge gravity approach gives rise to the nontrivial geometrical structures on the spacetime itself.

The convenient general framework for the discussion of the gauge gravity models is provided by the metric-affine gravity (MAG) theory [3]. This field-theoretic scheme contains as particular cases the models based on the group of translations (teleparallel gravity), on the local Poincaré group (Einstein–Cartan theory is perhaps the most well known), on the conformal or de Sitter group, and many other physically interesting theories. The gauge group of MAG is the local general affine group, a semidirect product of the group of translations on the general linear group $\mathrm{GL}(4,R)$. The corresponding gravitational gauge potentials are the metric, coframe and connection:

$$g_{\alpha\beta}\,,\qquad h_i^\alpha\,,\qquad \Gamma_{i\beta}{}^\alpha\,. \tag{1}$$

Our notation and conventions are as in [3]. In particular, the Latin indices label the holonomic components with respect to a natural coordinate frame ∂_i, whereas the Greek indices denote the anholonomic components with respect to an arbitrary frame $e_\alpha = h^i_\alpha \partial_i$. Alternatively, especially when the spinor fields are absent, one can use as the fundamental gravitational field variables the holonomic metric and connection. They are related to (1) as

$$g_{kl} = g_{\alpha\beta} h_k^\alpha h_l^\beta\,,\qquad \Gamma_{kj}{}^i = h^i_\alpha \Gamma_{k\beta}{}^\alpha h_j^\beta + h^i_\alpha \partial_k h_j^\alpha\,. \tag{2}$$

The gravitational field strengths in the metric-affine approach are the curvature

$$R_{kli}{}^j = \partial_k \Gamma_{li}{}^j - \partial_l \Gamma_{ki}{}^j + \Gamma_{kn}{}^j \Gamma_{li}{}^n - \Gamma_{ln}{}^j \Gamma_{ki}{}^n\,, \tag{3}$$

the torsion

$$T_{ij}{}^k = \Gamma_{ij}{}^k - \Gamma_{ji}{}^k\,, \tag{4}$$

and the nonmetricity

$$Q_{ikl} = -\nabla_i g_{kl} = -\partial_i g_{kl} + g_{jl}\Gamma_{ik}{}^j + g_{kj}\Gamma_{il}{}^j\,. \tag{5}$$

2. Model of a test particle

The Riemannian geometry arises as a special case of a general metric-affine spacetime when the gravitational variables satisfy the conditions of the vanishing torsion and nonmetricity, ${T_{ij}}^k = 0$, $Q_{ijk} = 0$. In order to detect the non-Riemannian structure, we need the matter which is characterized not only by the mass (energy-momentum) but also by an additional gravitational charge known as *hypermomentum*.

Here we describe the simplest model of a test particle with hypermomentum. It naturally generalizes the model of a spinning particle [4]. Such a particle is a physical point with an attached frame θ^k_a, where the index $a = 0, 1, 2, 3$ labels its legs. This frame is different from the gravitational variable h^k_α, hence a different notation. This material frame is not orthonormal, moreover in accordance with the affine gauge approach the scalar product of the frame's vectors is itself a dynamical variable in the theory, $g_{kl}\theta^k_a\theta^l_b = g_{ab}$. Like in the model of a spinning particle, [4], the material frame θ^k_a is firmly attached to the "body" of a particle. However, while for particle with spin its "body" is rigid, for a particle with hypermomentum the "body" is elastic, and the variable internal metric g_{ab} describes its possible deformations (along with the usual rigid rotations). Let t be an evolution parameter, then the motion of a particle with such an internal structure is described by the functions

$$x^k(t)\,, \qquad \theta^k_a(t)\,, \qquad g_{ab}(t)\,. \tag{6}$$

The internal metric is nondegenerate, its inverse is denoted by g^{ab}. The material frame is also nondegenerate and has the inverse θ^a_k, with $\theta^a_k\theta^k_b = \delta^a_b$.

The equations of motion can be derived for a large class of models with an unspecified Lagrangian function $L = L(x^k, \dot{x}^k, \theta^k_a, \dot{\theta}^k_a, g_{ab}, \dot{g}_{ab})$. We assume that L is a scalar with respect to the arbitrary coordinate transformations and with respect to the global general linear transformations

$$\theta^k_a \to \Lambda^b_a\theta^k_b\,, \qquad g_{ab} \to \Lambda^c_a\Lambda^d_b g_{cd}\,, \tag{7}$$

where Λ^a_b is an arbitrary 4×4 real nondegenerate matrix. Under these conditions, one can demonstrate that, when an elastic particle couples minimally to the gravitational field, the Lagrangian is a scalar function of three tensor arguments

$$L = L(v^k,\ {W_l}^k,\ g_{kl})\,. \tag{8}$$

Here $v^k := dx^k/dt$ is the velocity and, denoting the covariant derivative along the world line by $\dot{} := v^k\nabla_k$, we introduce

$${W_l}^k := \theta^a_l\dot{\theta}^k_a\,. \tag{9}$$

This tensor is a direct generalization of the angular velocity. It describes not only a rotation of a frame attached, but also a deformation of the "body".

We *define* (*cf.* [4]) the momentum P_k, the hypermomentum $J^k{}_l$, and the symmetric stress I^{kl} as the variational derivatives

$$P_k := \frac{\delta L}{\delta v^k}\,, \qquad J^k{}_l := \frac{\delta L}{\delta W_k{}^l}\,, \qquad I^{kl} := 2\frac{\delta L}{\delta g_{kl}}\,. \tag{10}$$

The condition that the Lagrangian function L is a scalar under the general coordinate transformations yields the identity

$$P_i\, v^j = I_i{}^j + W_i{}^k J^j{}_k - W_k{}^j J^k{}_i\,. \tag{11}$$

The symmetric part of (11) in fact determines the symmetric stress in terms of momentum and hypermomentum.

From the variation of the action with respect to the coordinate $x^i(t)$ and the material coframe $\theta^k_a(t)$ we find the equations of motion:

$$\dot{P}_i \;=\; T_{ij}{}^k v^j P_k + R_{ijk}{}^l v^j J^k{}_l - \frac{1}{2}Q_{ijk}(P^j v^k - W^{jl}J^k{}_l + W_l{}^k J^{jl})\,, \tag{12}$$

$$\dot{J}^i{}_j \;=\; W_k{}^i J^k{}_j - W_j{}^k J^i{}_k\,. \tag{13}$$

This system extends the Frenkel–Mathisson equations of motion [5–9] of a spinning particle to the most general case of dynamics of matter with hypermomentum in the non-Riemannian spacetime manifolds.

The model of an elastic particle can be generalized to a theory of a fluid with hypermomentum, the so-called hyperfluid [10].

3. General analysis of the equations of motion: conservation laws

The dynamical matter currents appear in the framework of the Noether theorem for the gauge symmetry of a theory and they are the sources of the corresponding gauge field. In MAG, these currents are determined by the variational derivatives of the matter Lagrangian $\mathcal{L}$ with respect to the gravitational gauge potentials. Thus we find the metric energy-momentum, the canonical energy-momentum and the canonical hypermomentum, respectively:

$$\sigma^{\alpha\beta} := 2\frac{\delta\mathcal{L}}{\delta g_{\alpha\beta}}\,, \qquad \Sigma_\alpha{}^k := \frac{\delta\mathcal{L}}{\delta h^\alpha_k}\,, \qquad \Delta^\beta{}_\alpha{}^k := \frac{\delta\mathcal{L}}{\delta \Gamma_{k\beta}{}^\alpha}\,. \tag{14}$$

The canonical currents for the hyperfluid [10] read (u^k is the 4-velocity)

$$\Sigma_\alpha{}^k = u^k P_\alpha - (h^k_\alpha - u^k u_\alpha)p\,, \qquad \Delta^\alpha{}_\beta{}^k = u^k J^\alpha{}_\beta\,. \tag{15}$$

When the pressure vanishes $p = 0$ (the state equation of a dust), these formulas reproduce the dynamical currents of the elastic test particle. The densities of the 4-momentum and hypermomentum $(P_\alpha, J^\alpha{}_\beta)$ thus correctly describe the generalized charge which is the source of the gravitational field in the gauge-theoretic scheme based on the general affine group.

The equations of motion of extended bodies in MAG (for preliminary results see [11]) can be derived along the general lines of the Mathisson–Papapetrou multipole expansion method which was successfully applied to the Poincaré gauge gravity in [12, 13]. At the center of this approach one puts the conservation laws (or, more precisely, the balance equations) of the energy-momentum $\Sigma_\alpha{}^k$ and of the hypermomentum $\Delta^\beta{}_\alpha{}^k$. The latter generalizes the conservation law of the angular momentum, see the discussion in [3] and more recently in [14].

The analysis within MAG [17] confirms the conclusion obtained previously in the Poincaré gauge gravity [12] that the bodies composed of the usual matter (without microstructure) cannot detect the non-Riemannian spacetime geometry. This result is of practical importance in connection with the current programs of space experiments, see the discussion in [15–17].

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REFERENCES

[1] A. Einstein, "Geometrie und Erfahrung", *Sitzungsber. Preuss. Akad. Wiss.* **1**, 123 (1921) [English translation see: *The Collected Papers of A. Einstein*, Princeton Univ. Press, Princeton and Oxford 2002, vol. 7, pp. 208–222].

[2] A. Trautman, *Einstein–Cartan Theory*, in: *Encyclopedia of Mathematical Physics*, Eds. J.P. Françoise, G.L. Naber, S.T. Tsou, Elsevier, Oxford 2006, vol. 2, pp. 189–195; `http://www.fuw.edu.pl/~amt/ect.pdf`.

[3] F.W. Hehl, J.D. McCrea, E.W. Mielke, Y. Ne'eman, *Phys. Rep.* **258**, 1 (1995).

[4] W. Kopczyński, *Phys. Rev.* **D34**, 352 (1986).

[5] J. Frenkel, *Z. Phys.* **37**, 243 (1926).

[6] M. Mathisson, *Acta Phys. Pol.* **6**, 163 (1937).

[7] J. Weyssenhoff, A. Raabe, *Acta Phys. Pol.* **9**, 7 (1947).

[8] I. Bailey, W. Israel, *Commun. Math. Phys.* **42**, 65 (1975).

[9] I. Bailey, *Ann. Phys. (USA)* **119**, 76 (1979).

[10] Yu.N. Obukhov, R. Tresguerres, *Phys. Lett.* **A184**, 17 (1993).

[11] Y. Ne'eman, F.W. Hehl, *Class. Quantum Grav.* **14**, 251 (1997).

[12] P.B. Yasskin, W.R. Stoeger, *Phys. Rev.* **D21**, 2081 (1980).

[13] K. Nomura, T. Shirafuji, K. Hayashi, *Prog. Theor. Phys.* **86**, 1239 (1991).

[14] Yu.N. Obukhov, G.F. Rubilar, *Phys. Rev.* **D74**, 064002 (2006); *Phys. Rev.* **D76**, 124030 (2007).

[15] Y. Mao, M. Tegmark, A. Guth, S. Cabi, *Phys. Rev.* **D76**, 104029 (2007).

[16] E. Flanagan, E. Rosenthal, *Phys. Rev.* **D75**, 124016 (2007).

[17] D. Puetzfeld, Yu.N. Obukhov, *Phys. Rev.* **D76**, 084025 (2007).