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HIGHLY RELATIVISTIC MOTIONS OF SPINNING PARTICLES ACCORDING TO MATHISSON EQUATIONS*

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The physical effects following from the Mathisson equations at the highly relativistic motions of a spinning test particle relative to a Schwarzschild mass are discussed. The corresponding numerical estimates are presented.

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1. Introduction

During 70 years the Mathisson equations [1] have being investigated by many authors with different intensity. The very fruitful years were 1970s [2–11]. There is an interesting remark in [5], p. 111: "*The simple act of endowing a black hole with angular momentum has led to an unexpected richness of possible physical phenomena. It seems appropriate to ask whether endowing the test body with intrinsic spin might not also lead to surprises*". In this context the question of importance is: can spin of a test particle essentially change its world line and trajectory? To answer this question it is useful to consider the Mathisson equations both in their traditional form and in the terms of the local (tetrad) quantities connected with the moving particle. The initial form of the Mathisson equations is [1]

$$\frac{D}{ds}\left(mu^\lambda + u_\mu\frac{DS^{\lambda\mu}}{ds}\right) = -\frac{1}{2}u^\pi S^{\rho\sigma}R^\lambda_{\pi\rho\sigma}\,, \tag{1}$$

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$$\frac{DS^{\mu\nu}}{ds} + u^{\mu} u_{\sigma} \frac{DS^{\nu\sigma}}{ds} - u^{\nu} u_{\sigma} \frac{DS^{\mu\sigma}}{ds} = 0\,, \tag{2}$$

where u^{λ} is the 4-velocity of a spinning particle, $S^{\mu\nu}$ is the antisymmetric tensor of spin, m and D/ds are, respectively, the mass and the covariant derivative with respect to the proper time s; $R^{\lambda}_{\pi\rho\sigma}$ is the Riemann curvature tensor of the spacetime. (Greek indices run $1, 2, 3, 4$ and Latin indices $1, 2, 3$.) Equations (1), (2) were supplemented by the condition [1]

$$S^{\mu\nu} u_{\nu} = 0\,, \tag{3}$$

which first was used in electrodynamics [12]. Later the condition

$$S^{\mu\nu} P_{\nu} = 0 \tag{4}$$

was introduced [2, 13], where

$$P^{\nu} = m u^{\nu} + u_{\mu} \frac{DS^{\nu\mu}}{ds}\,. \tag{5}$$

Concerning the physical meaning of conditions (3) and (4) see, *e.g.,* [14].

Besides $S^{\mu\nu}$, the 4-vector of spin s_{λ} is also used in the literature, where by definition [3]

$$s_{\lambda} = \frac{1}{2}\sqrt{-g}\,\varepsilon_{\lambda\mu\nu\sigma} u^{\mu} S^{\nu\sigma}\,, \tag{6}$$

(g is the determinant of the metric tensor) with the relation

$$s_{\lambda} s^{\lambda} = \frac{1}{2} S_{\mu\nu} S^{\mu\nu} = S_0^2\,, \tag{7}$$

where S_0 is the constant of spin.

2. Mathisson equations in representation of Ricci's coefficients of rotation

For transformation of equations (1), (2) under condition (3) we use the relations for the comoving orthogonal tetrads $\lambda^{(\nu)}_{\mu}$:

$$dx^{(i)} = \lambda^{(i)}_{\mu} dx^{\mu} = 0\,, \qquad dx^{(4)} = \lambda^{(4)}_{\mu} dx^{\mu} = ds \tag{8}$$

(here indices of the tetrad are placed in the parentheses). For convenience, we choose the first local coordinate axis as oriented along the spin, then

$$\begin{aligned} s_{(1)} \neq 0\,, \qquad & s_{(2)} = 0\,, \\ s_{(3)} = 0\,, \qquad & s_{(4)} = 0\,, \end{aligned} \tag{9}$$

and $|s_{(1)}| = S_0$.

By (3), (8), (9) it follows from (2) that $\gamma_{(i)(k)(4)} = 0$, *i.e.,* the known condition for the Fermi transport, where $\gamma_{(\alpha)(\beta)(\gamma)}$ are Ricci's coefficients of rotation. From equations (1) one can find

$$m\gamma_{(1)(4)(4)} + s_{(1)}R_{(1)(4)(2)(3)} \; = \; 0\,, \tag{10}$$

$$m\gamma_{(2)(4)(4)} + s_{(1)}(R_{(2)(4)(2)(3)} - \dot{\gamma}_{(3)(4)(4)}) \; = \; 0\,, \tag{11}$$

$$m\gamma_{(3)(4)(4)} + s_{(1)}(R_{(3)(4)(2)(3)} + \dot{\gamma}_{(2)(4)(4)}) \; = \; 0\,, \tag{12}$$

where a dot denote the usual derivatives with respect to s. The Ricci coefficients of rotation $\gamma_{(i)(4)(4)}$ have the physical meaning as the components $a_{(i)}$ of the 3-acceleration of a spinning particle relative to geodesic free fall as measured by the comoving observer.

In the linear in spin approximation equations (10)–(12) can be written as [15]

$$\gamma_{(i)(4)(4)} \equiv a_{(i)} = -\frac{s_{(1)}}{m} R_{(i)(4)(2)(3)}\,. \tag{13}$$

It is known that within the framework of this approximation the physical consequences following from equations (1), (2) coincide under conditions (3) and (4) [16, 17].

By definition of the gravitoelectric $E^{(i)}_{(k)}$ and the gravitomagnetic $B^{(i)}_{(k)}$ components we have [18]

$$E^{(i)}_{(k)} \; = \; R^{(i)(4)}{}_{(k)(4)}\,, \tag{14}$$

$$B^{(i)}_{(k)} \; = \; -\frac{1}{2} R^{(i)(4)}{}_{(m)(n)} \varepsilon^{(m)(n)}{}_{(k)}\,. \tag{15}$$

So, according to (13), (15) the acceleration $a_{(i)}$ is determined by $B^{(i)}_{(k)}$.

3. Case of a Schwarzschild metric

Let us consider equation (13) for Schwarzschild metric in standard coordinates $x^1 = r$, $x^2 = \theta$, $x^3 = \varphi$, $x^4 = t$. The motion of an observer relative to Schwarzschild's mass M can be described by the orthonormal frame $\lambda^{(\nu)}_{\mu}$. For expediency and without loss of generality we assume that the first tetrad axis is perpendicular to plane determined by the direction of observer motion and the radial direction ($\theta = \pi/2$), and the second axis coincides with the direction of motion. Then the non-zero components of $B^{(i)}_{(k)}$ are [19]

$$B^{(1)}_{(2)} = B^{(2)}_{(1)} = \frac{3Mu_{\parallel}u_{\perp}}{r^3\sqrt{u_4u^4 - 1}} \left(1 - \frac{2M}{r}\right)^{-1/2} ,$$

$$B^{(1)}_{(3)} = B^{(3)}_{(1)} = \frac{3Mu_\perp^2 u^4}{r^3\sqrt{u_4 u^4 - 1}}\left(1 - \frac{2M}{r}\right)^{1/2}\,, \tag{16}$$

where $u_\parallel = \dot r$ is the radial component of the observer 4-velocity , $u_\perp = r\dot\varphi$ is the tangential component.

Let a spinning particle be comoving with the observer and its spin is oriented along the first tetrad axis. By (15), (16) relation (13) can be written as

$$a_{(i)} = \frac{s_{(1)}}{m} B^{(1)}_{(i)}\,. \tag{17}$$

From (16), (17) it is easy to see that the acceleration $a_{(i)}$ is not equal to 0 if and only if $u_\perp \neq 0$, *i.e.*, is caused by the gravitational spin–orbit interaction. (The gravitational spin–orbit and spin–spin interactions in the post-Newtonian approximation were investigated in [4].) By (16), (17) we have

$$|\vec a_{\rm s.-o.}| = \frac{M}{r^2}\frac{3s_{(1)}|u_\perp|}{mr}\sqrt{1 + u_\perp^2}\,, \tag{18}$$

where $|\vec a_{\rm s.-o.}| \equiv \sqrt{a^2_{(1)} + a^2_{(2)} + a^2_{(3)}}$ is the absolute value of the gravitational spin–orbit acceleration. While investigating possible effects of spin on the particle's motion it is necessary to take into account the condition for a spinning test particle [4]

$$\varepsilon \equiv \frac{|s_{(1)}|}{mr} \ll 1\,. \tag{19}$$

According to (18), (19), the two limiting cases are essentially different in their physical consequences: (1) at low velocity, when $|u_\perp| \ll 1$, we have $|\vec a_{\rm s.-o.}| \ll M/r^2$, where M/r^2 is the Newtonian value of the free fall acceleration; (2) at high velocity, when $|u_\perp| \gg 1$, for any small ε we can indicate such sufficiently large value $|u_\perp|$ for which the value $|\vec a_{\rm s.-o.}|$ is of order M/r^2. That is, in the second case the motion of a spinning particle essentially differs from the geodesic motion [15]. We stress that this conclusion is obtained from the point of view of the comoving observer. Is the similar conclusion valid for an observer which, for example, is at rest relative to the Schwarzschild mass? To answer this question it is necessary to investigate the corresponding solutions of equations (1), (2).

The interesting partial solutions of equations (1), (2) in a Schwarzschild spacetime were studied in [19, 20]. Namely, it is shown that the circular highly relativistic orbits of a spinning particle exist in the small neighborhood of the value $r = 3M$ (both for $r > 3M$ and $r < 3M$) with

$$|u_\perp| = \frac{3^{1/4}}{\sqrt{\varepsilon}} \tag{20}$$

(*i.e.,* according to (19), (20) we have $u_\perp^2 \gg 1$). These orbits differ from the highly relativistic geodesic circular orbits, which exist only for $r > 3M$. Besides, it is known that a particle without spin and with non-zero mass of any velocity close to the velocity of light, starting in the tangential direction from the position $r = 3M$, will fall on Schwarzschild's horizon within a finite proper time, whereas a spinning particle will remain indefinitely on the circular orbit $r = 3M$ due to the interaction of its spin with the gravitational field, *i.e.,* this interaction compensates the usual (geodesic) attraction.

The highly relativistic circular orbits determined by (20) are practically common for equations (1), (2) at conditions (3) and (4) [20]. Outside the small neighborhood of $r = 3M$, for $2M < r < 3M$, equations (1), (2) admit circular highly relativistic orbits as well, however, only under condition (3) [14, 20]. Similarly to (20), the value $|u_\perp|$ on these orbits is of order $1/\sqrt{\varepsilon}$.

Some non-circular essentially non-geodesic orbits with the initial values of $|u_\perp|$ of order $1/\sqrt{\varepsilon}$ were computed in [14].

4. Conclusions and numerical estimates

So, if the tangential component of the particle's velocity is of order $1/\sqrt{\varepsilon}$, its spin can essentially deviate the particle's trajectory from the geodesic line, both from the point of view of the comoving observer and from the point of view of an observer which is at rest relative to the Schwarzschild mass. (By (18), if $|u_\perp|$ is of order $1/\sqrt{\varepsilon}$, the acceleration $|\vec{a}_{\mathrm{s.-o.}}|$ is of order M/r^2.)

The effect of the considerable space separation of spinning and non-spinning particles takes place for the short time: less than one or two revolutions of the spinning particles around a Schwarzschild mass the difference of the radial coordinates Δr becomes comparable with the initial radial coordinate [14].

The existence of highly relativistic circular orbits of a spinning particle in a Schwarzschild field, which differ from geodesic circular orbits, perhaps, can be discovered in the synchrotron radiation of protons or electrons.

Let us estimate the value $\varepsilon = |S_0|/Mm$ for protons and electrons in the cases when the Schwarzschild source is (1.) a black hole of mass that is equal to three of the Sun's mass, and (2.) a massive black hole of mass that is equal to 10^8 of the Sun's mass. In the first case, taking into account the numerical values of S_0, M, m in the system used, we have for protons and electrons correspondingly $\varepsilon_p \approx 2 \times 10^{-20}$, $\varepsilon_e \approx 4 \times 10^{-17}$, whereas in the second case $\varepsilon_p \approx 7 \times 10^{-28}$, $\varepsilon_e \approx 10^{-24}$. Because for the motion on above considered orbits the spinning particles must possess the velocity corresponding to relativistic Lorentz γ-factor of order $1/\sqrt{\varepsilon}$, in the first case we obtain $\gamma_p \approx 7 \times 10^9$, $\gamma_e \approx 2 \times 10^8$, and in the second case $\gamma_p \approx 4 \times 10^{13}$, $\gamma_e \approx 10^{12}$.

As we see, in the case of a massive black hole the necessary values of γ_p and γ_e are too high even for extremely relativistic particles from the cosmic rays. Whereas if the Schwarzschild source is a black hole of mass that is equal to 3 of the Sun's mass, some particles may move on the circular orbits considered above. Analysis of a concrete model, closer to the reality, remains to be carried out. Here we point out that by the known general relationships for the electromagnetic synchrotron radiation in the case of protons or electrons on the considered circular orbits we obtain the values from the gamma-ray range.

The results of investigation of the highly relativistic considerable non-geodesic motions of a spinning test particle according to the Mathisson equations stimulate the analysis of the corresponding quantum states which are described by the Dirac equations [19].

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