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CLASSICAL AND QUANTUM SPINS IN CURVED SPACETIMES*

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A comparative analysis of the Mathisson–Papapetrou and Pomeransky–Khriplovich equations is presented. Motion of spinning particles and their spins in gravitational fields and noninertial frames is considered. The angular velocity of spin precession defined by the Pomeransky–Khriplovich equations depends on the choice of the tetrad. The connection of such a dependence with the Thomas precession is established. General properties of spin interactions with gravitational fields are discussed. It is shown that dynamics of classical and quantum spins in curved spacetimes is identical. A manifestation of the equivalence principle in an evolution of the helicity is analyzed.

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1. Introduction

Spin dynamics in curved spacetimes is an important part of spin physics. Spin effects in gravitational fields and noninertial frames are important not only for particles but also for gyroscopes and even celestial bodies. Many such effects can be discovered and investigated in cosmic experiments. Therefore, a necessary theoretical description of the spin dynamics in curved spacetimes should be carries out.

Pioneering calculations of the spin effects in gravitational fields were made soon after the creation of the general relativity [1–3]. However, an investigation of mutual influence of particle and spin motion in curved spacetimes was started from the excellent work by Mathisson [4]. Another investigation of this problem was performed by Pomeransky and Khriplovich [5].

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We present a comparative analysis of different equations of motion of spinning particles and their spins, discuss their connection with the equivalence principle, and investigate specific effects. In the next section, we introduce the Mathisson–Papapetrou equations (MPE). In Sec. 3, we briefly discuss general properties of spin interactions with gravitational fields. Next section is devoted to the comparison between the MPE and Pomeransky–Khriplovich equations (PKE). Equations of spin motion in stationary spacetimes are discussed in Sec. 5. In Sec. 6, the spin effects in classical and quantum gravity are compared. A manifestation of the equivalence principle in an evolution of the helicity in gravitational fields and noninertial frames is analyzed in Sec. 7. Finally, in Sec. 8 we discuss previously obtained results and summarize the main results of the work.

Throughout the work tetrad indices are designated by first Latin letters. Greek indices and other Latin indices run $0, 1, 2, 3$ and $1, 2, 3$, respectively. The metric signature $(+, -, -, -)$ is chosen. We use the designations $[\dots, \dots]$ and $\{\dots, \dots\}$ for commutators and anticommutators, respectively. We use the term "tetrad vector" for vectors formed from tetrad components.

2. Mathisson–Papapetrou equations

The famous MPE first found by Mathisson [4] and then rediscovered by Papapetrou [6] describe dynamics of a classical spinning particle and their spin in curved spacetimes. All multipole moments of an extended body in a gravitational field was taken into account by Dixon [7]. The explicit form of the MPE is

$$\frac{Dp^\mu}{ds} = -\frac{1}{2} R^\mu_{\nu\alpha\beta} u^\nu S^{\alpha\beta}\,, \tag{1}$$

$$\frac{DS^{\mu\nu}}{ds} = p^\mu u^\nu - p^\nu u^\mu\,, \tag{2}$$

where u^ν and p^ν are the four-velocity and four-momentum of the spinning particle, respectively, $R^\mu_{\nu\alpha\beta}$ is the Riemann curvature tensor of the spacetime, and $D/(ds)$ means the covariant derivative with respect to the interval ds.

These equations should be supplemented with the condition [4, 8]

$$S^{\mu\nu} u_\nu = 0\,, \tag{3}$$

or [7, 9, 10]

$$S^{\mu\nu} p_\nu = 0\,. \tag{4}$$

The MPE characterize the pole–dipole approximation, when multipole moments of higher orders are neglected. These equations predict the mutual influence of particle and spin motion. In particular, spinning particles undergo an additional force which is similar to the Stern–Gerlach force in electrodynamics. As a result, spinning particles do not move on geodesics in curved spacetimes.

In a zero approximation, one can neglect the mutual influence of particle and spin motion. In this approximation, the spin tensor is parallel transported in the spacetime and the MPE take the form

$$\frac{Dp^\mu}{ds} = 0\,, \tag{5}$$
$$\frac{DS^{\mu\nu}}{ds} = 0\,, \tag{6}$$
$$p^\mu = mcu^\mu\,, \tag{7}$$

where m is the mass of the particle.

3. General properties of spin interactions with gravitational fields

General properties of interactions of classical spin with gravitational fields can be obtained, when the mutual influence of particle and spin motion is neglected.

The curvature of spacetime conditions a precession of moving spinning particles and gyroscopes (geodetic effect [1, 2]). Additional rotation of the spin in a gravitational field of a rotating body is caused by the frame dragging (Lense–Thirring effect [3]). This effect results in appearing an additional acceleration similar to the Coriolis one and an additional precession of satellite orbits and the spin. Similar effects take place in a rotating frame. In the nonrelativistic approximation, resulting motion of the spin is given by [11]

$$\frac{d\boldsymbol{S}}{dt} = \boldsymbol{\Omega}\times\boldsymbol{S}\,, \qquad \boldsymbol{\Omega} = \frac{3GM}{2c^2r^3}(\boldsymbol{r}\times\boldsymbol{v}) + \frac{G}{c^2r^3}\left[\frac{3(\boldsymbol{J}\cdot\boldsymbol{r})\boldsymbol{r}}{r^2} - \boldsymbol{J}\right], \tag{8}$$

where M and $\boldsymbol{J}$ are the mass and angular moment of the central body and $\boldsymbol{v}$ is the velocity of spinning test particle. As was mentioned in Ref. [11], Eq. (8) is consistent with approximate Mathisson–Papapetrou equation (6).

Eq. (8) results in the conclusion that an anomalous gravitomagnetic moment (AGM) and a gravitoelectric dipole moment (GDM) are equal to zero. Indeed, the angular velocity of spin rotation depends on neither the mass nor the spin. Therefore, the relation between the torque $d\boldsymbol{S}/dt$ and the spin $\boldsymbol{S}$ is the same for all particles/gyroscopes. This is an explicit manifestation of the absence of the AGM. The equality of angular velocities of spin rotation of all particles in the stationary spacetime created by the rotating body is another manifestation of the absence of the AGM. Evidently, the latter property is also valid for spinning particles (gyroscopes) in the rotating frame. The absence of the GDM results from the fact that the spin of particle at rest does not interact with a static gravitational field if tidal forces can be neglected.

It is not evident whether the above mentioned conclusion remains valid for quantum particles. This problem was much-discussed (see Ref. [12] and references therein). Nevertheless, an explicit solution of the problem was obtained many years ago by Kobzarev and Okun [13]. In this work, gravitational interactions of spin-1/2 particles have been considered and important relations for gravitational form factors at zero momentum transfer have been derived. It was proved that gravitational and inertial masses are equal ($f_1 = g_1 = 1$), any gravitomagnetic moment is "normal" (*i.e.* the AGM is equal to zero ($f_2 = 1$)), and the GDM is equal to zero ($f_3 = 0$). The generalization of these properties to arbitrary-spin particles was made by Teryaev [14].

The absence of the GDM results in the absence of spin-gravity coupling

$$W \sim \boldsymbol{g} \cdot \boldsymbol{S}, \tag{9}$$

where $\boldsymbol{g}$ is the gravitational acceleration.

The relations obtained by Kobzarev and Okun lead to equal frequencies of precession of quantum (spin) and classical (orbital) angular momenta and the preservation of helicity of Dirac particles in gravitomagnetic fields (*i.e.* the fields defined by the components g_{i0} of the metric tensor, see Ref. [14] and references therein). Any reference frame characterized by the nonzero g_{i0} possesses these properties. In particular, one can mention the gravitational field of massive rotating body and the noninertial rotating frame.

Thus, the general properties of spin interactions with gravitational fields are the same for classical and quantum particles.

Similarity of Eqs. (5) and (6) conditions conformity of particle and spin dynamics in the general relativity. The equality of angular velocities of spin rotation of all particles is similar to the independence of particle accelerations in curved spacetimes on their masses. Therefore, the above discussed general properties of spin interactions with gravitational fields can be considered as the manifestations of the equivalence principle in spin-gravity interactions (see Ref. [14]).

4. Comparison between Mathisson–Papapetrou and Pomeransky–Khriplovich equations

There are two possible methods used for the derivation of the MPE and PKE [5]. First method consists in a search for appropriate covariant equations. This method was utilized for the derivation of classical equations of spin motion in electromagnetic fields [15–17]. The Thomas–Bargmann–Michel–Telegdi (T-BMT) equation [15, 16] and the Good–Nyborg equation (GNE) [17] describe spin dynamics in uniform and nonuniform electromagnetic fields, respectively. The same method was applied by Mathisson [4] and Papapetrou [6] for obtaining equations of spin motion in curved spacetimes.

Second method consists in a deduction of equations on the basis of some physical principles without an attempt to obtain covariant final equations. Pomeransky and Khriplovich [5] used this method for the derivation of equations of spin motion in electromagnetic fields with allowance for terms of the first and second orders in spin. This method is based on the fact that the three-component spin defined in a particle rest frame is a noncovariant quantity [5]. The use of covariant equations may be possible if coordinates are redefined [5]. The validity of the "noncovariant" [5] approach was confirmed with a comparison between the GNE, PKE for the electromagnetic field [5], and the equation deduced in Refs. [18, 19] from the Hamiltonian for spin-1 particles in the electromagnetic field [20]. It was shown [18, 19] that the Foldy–Wouthyusen (FW) transformation followed by the semiclassical transition results in the equation of spin motion which agrees with the PKE but contradicts to the GNE. This is an indirect confirmation of the noncovariant approach which was also used [5] for description of gravitational interactions.

To find a connection between MPE and PKE, we use the results obtained by Chicone, Mashhoon, and Punsly [21]. The relation between the four-momentum and four-velocity has the form

$$p^{\mu} = mcu^{\mu} + E^{\mu}\,, \tag{10}$$

where the order of magnitude of E^{μ} is given by

$$E^{\mu} \sim \frac{1}{mc} S^{\mu\nu} \frac{Dp_{\nu}}{ds}\,. \tag{11}$$

The additional four-force is $DE^{\mu}/d\tau$, where τ is the proper time. This four-force is of the second order in the spin [21]. The approximate equation of the first order in the spin resulting from Eq. (1) is [21]

$$mc\frac{Du^{\mu}}{ds} = -\frac{1}{2} R^{\mu}_{\nu\alpha\beta} u^{\nu} S^{\alpha\beta}\,. \tag{12}$$

Eq. (8) and the PKE unambiguously show that the spin dynamics depends on derivatives of the metric tensor. The right hand side of Eq. (2) defined by Eqs. (1), (10), (11) is of the order of

$$p^{\mu}u^{\nu} - p^{\nu}u^{\mu} \sim \frac{1}{mc} R_{\lambda\alpha\beta\gamma} u^{\alpha} u^{\nu} S^{\mu\lambda} S^{\beta\gamma}\,. \tag{13}$$

This quantity is much less than terms defining the spin motion in the PKE [5], when the weak-field approximation is used. In this approximation

$$|g_{\mu\nu} - \eta_{\mu\nu}| = |h_{\mu\nu}| \ll 1\,, \tag{14}$$

where the tensor $\eta_{\mu\nu}$ characterizes the Minkowski spacetime. In addition, the right hand side in Eq. (13) is of the second order in the spin tensor. Therefore, the correction to Eq. (6) is rather small. When terms of the first order in the spin are retained, the MPE reduce to Eqs. (6), (12).

To derive the corresponding equation for the spin (pseudo)vector S^μ in the same approximation, we can use the known connection between the spin vector and the spin tensor [22] and Eq. (12). When only terms of the first order in the spin are taken into account, the equation for the spin vector is given by

$$\frac{DS^\mu}{ds} = 0\,. \tag{15}$$

This is the initial equation used by Pomeransky and Khriplovich [5]. Therefore, we can conclude that the spin dynamics predicted by the MPE and PKE is the same in the first-order approximation in the spin. A possible difference between two approaches can be caused by second-order terms in the spin (including quadrupole interactions). Such terms was calculated in Ref. [5] in the framework of quantum theory. In the present work, we confine ourselves to the discussion of first-order spin effects.

Eq. (15) should be supplemented with the orthogonality condition

$$S^\mu u_\mu = 0\,. \tag{16}$$

The method developed by Pomeransky and Khriplovich [5] is based on the equations of motion of particles and their spins in the zero approximation [Eqs. (5) and (15), respectively]. In Ref. [5], the former equation was written for the four-velocity and the tetrad formalism was used. The equations for the tetrad components of the four-velocity $u^a = e^a_\mu u^\mu$ and the four-component spin $S^a = e^a_\mu S^\mu$ are [5]

$$\frac{du_a}{ds} = \gamma_{abc} u^b u^c\,, \tag{17}$$

$$\frac{dS_a}{ds} = \gamma_{abc} S^b u^c\,. \tag{18}$$

Here e^a_μ is the vierbein and

$$\gamma_{abc} = e_{a\mu;\nu} e^\mu_b e^\nu_c = -\gamma_{bac}$$

are the Ricci rotation coefficients [23].

Evidently, these equations are similar to the equations of motion of particles with zero anomalous magnetic moment ($g = 2$) and their spins in an electromagnetic field:

$$\frac{du_\mu}{d\tau} = \frac{e}{mc} F_{\mu\nu} u^\nu \,, \qquad \frac{dS_\mu}{d\tau} = \frac{e}{mc} F_{\mu\nu} S^\nu \,, \tag{19}$$

where $F_{\mu\nu}$ is the electromagnetic field tensor.

Therefore, the following correspondence takes place [5]:

$$\frac{e}{mc^2} F_{ab} \leftrightarrow \gamma_{abc} u^c \,. \tag{20}$$

The antisymmetric electromagnetic field tensor has six independent components and is defined by the electric and magnetic fields:

$$F_{ab} \leftrightarrow (\boldsymbol{E}, \boldsymbol{B}) \,. \tag{21}$$

One can similarly define the gravitoelectric and gravitomagnetic fields:

$$\frac{e}{mc} \boldsymbol{E} \leftrightarrow \boldsymbol{\mathcal{E}} \,, \qquad \frac{e}{mc} \boldsymbol{B} \leftrightarrow \boldsymbol{\mathcal{B}} \,, \qquad c\gamma_{abc} u^c \leftrightarrow (\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{B}}) \,. \tag{22}$$

An important difference between the electromagnetic and gravitational interactions consists in the fact that $\gamma_{abc} u^c$ is not a tensor. Explicit expressions for the gravitoelectric and gravitomagnetic fields are (see Ref. [5])

$$\mathcal{E}_{\hat{i}} = c\, \gamma_{0ic} u^c \,, \qquad \mathcal{B}_{\hat{i}} = -\frac{c}{2}\, e_{\hat{i}\hat{k}\hat{l}} \gamma_{klc} u^c \,, \tag{23}$$

where $e_{\hat{i}\hat{k}\hat{l}}$ is the antisymmetric tensor with spatial components. To avoid misleading coincidences, zero and spatial tetrad indexes are marked with hats (except for the Ricci rotation coefficients).

The comparison with the T-BMT equation [15, 16] allows to obtain the angular velocity of spin precession. Pomeransky and Khriplovich introduced the three-component spin (pseudo)vector $\boldsymbol{S}$ and obtained the exact equation of its precession [5]

$$\frac{d\boldsymbol{S}}{dt} = \boldsymbol{\Omega} \times \boldsymbol{S} \,, \qquad \Omega_{\hat{i}} = c e_{\hat{i}\hat{k}\hat{l}} \left(\frac{1}{2}\, \gamma_{klc} + \frac{u^{\hat{k}}}{u^{\hat{0}} + 1} \gamma_{0lc} \right) \frac{u^c}{u^0} \,, \tag{24}$$

that is equivalent to

$$\boldsymbol{\Omega} = \frac{1}{u^0} \left[-\hat{\boldsymbol{\mathcal{B}}} + \frac{\hat{\boldsymbol{u}} \times \hat{\boldsymbol{\mathcal{E}}}}{u^{\hat{0}} + 1} \right] \,. \tag{25}$$

The factor $1/u^0$ is caused by the transition to the differentiation with respect to the world time t. The definitions of $\boldsymbol{\Omega}$ in Refs. [5,25] and the present work differ in sign. When the differentiation is performed with respect to the tetrad time ($d\hat{t} = u^{\hat{0}}dt/u^0$), Eq. (25) coincides with the T-BMT equation for Dirac particles ($g = 2$). The gravitoelectric and gravitomagnetic fields are defined via their tetrad components. The quantity $\boldsymbol{\Omega}$ characterizes the spin precession in the world frame, while the spin $\boldsymbol{S}$ is defined in the particle rest frame. In this connection, the dependence of $\boldsymbol{\Omega}$ on the choice of the tetrad must result from a change of the particle rest frame.

For a Schwarzschild metric, the exact expresson for $\boldsymbol{\Omega}$ was obtained in Ref. [25].

The corresponding equation of particle motion has the form

$$\frac{d\hat{\boldsymbol{u}}}{dt} = \frac{u^{\hat{0}}}{u^0}\left(\hat{\boldsymbol{\mathcal{E}}} + \frac{\hat{\boldsymbol{u}} \times \hat{\boldsymbol{\mathcal{B}}}}{u^{\hat{0}}}\right), \qquad \frac{du^{\hat{0}}}{dt} = \frac{\hat{\boldsymbol{\mathcal{E}}} \cdot \hat{\boldsymbol{u}}}{u^0}. \tag{26}$$

When the differentiation is performed with respect to the tetrad time, Eq. (26) coincides with the Lorentz equation. Eqs. (17), (26) describe the particle motion along geodesic lines.

Definition (23) of the gravitoelectric and gravitomagnetic fields significantly differs from the usual one. In particular, the Pomeransky–Khriplovich gravitomagnetic field is nonzero even for a static metric.

There is a one-to-one correspondence between the angular velocity of precession of the three-component spin and spin-dependent terms in classical [5] and quantum [24] Lagrangians and Hamiltonians. To derive spin-dependent corrections to classical Lagrangians, Poisson brackets was used in Ref. [5]. When classical and quantum expressions for $\boldsymbol{\Omega}$ coincide, the spin-dependent terms in classical and quantum Lagrangians/Hamiltonians derived with the Poisson brackets and commutators, respectively, are also the same. These terms are given by

$$\mathcal{L} = \mathcal{L}_0 + \boldsymbol{\Omega} \cdot \boldsymbol{S}, \qquad \mathcal{H} = \mathcal{H}_0 - \boldsymbol{\Omega} \cdot \boldsymbol{S}, \tag{27}$$

where $\mathcal{L}_0$ and $\mathcal{H}_0$ define sums of spin-independent terms. It will be shown below that Eq. (24) agrees with corresponding equations derived in the framework of quantum theory. As a result, classical Lagrangians and Hamiltonians defined by Eqs. (24) and (27) must agree with corresponding quantum Hamiltonians. Thus, the PKE are consistent with the quantum gravity at least in the first-order approximation in the spin.

Influences of the spin on a particle trajectory in a gravitational field predicted by the MPE and PKE significantly differ [5, 25]. It was stated in Refs. [5, 25, 26] that the MPE are not consistent with Eq. (8) describing the geodetic effect (gravitational spin-orbit interaction). In the Pomeransky–Khriplovich approach, the consistence of motion of particles and their spins results from Eq. (27).

5. Equations of spin motion in stationary spacetimes

While MPE (6) is equivalent to PKE (15), general equation of spin motion (24) was obtained only in the framework of the Pomeransky–Khriplovich approach. However, the Pomeransky–Khriplovich method needs to be grounded. Eqs. (18) and (19) describing the motion of the four-component spin vector in gravitational and electromagnetic fields, respectively, are very similar. However, there exists an important difference between the corresponding equations for the three-component spin vector. Since the latter vector is defined in the particle rest frame, the spatial components of the four-velocity satisfying Eq. (16) are equal to zero in this frame. Such a definition of the velocity is ambiguous because this quantity can be characterized by covariant, contravariant, and tetrad vectors. A definite choice can be made due to a local Lorentz invariance. The spacetime metric tensor locally has the Minkowski form $\eta_{\mu\nu}$ of special relativity in any freely-falling reference frame including the particle rest frame (see Ref. [27]). Tetrad components of any vector are similar to components of vectors in a flat spacetime. In particular, covariant and contravariant tetrad components of vectors are equal up to sign. Since the spacetime interval in tetrad coordinates takes the Minkowski form, tetrad reference frames are flat and correspond to local Lorentz frames.

The particle velocity is zero and the spacetime is flat in the particle rest frame. In this frame, just spatial tetrad components of the particle velocity are zero ($\hat{\boldsymbol{u}} = 0$). It is natural because any observer carries a tetrad frame (see Ref. [28]). Corresponding covariant and contravariant components (u^i and u_i, respectively) can be nonzero. Since the orthogonality condition can be written in the form

$$S^a u_a = 0\,, \tag{28}$$

the three-component spin is composed by spatial components of the four-component tetrad spin at condition that $\hat{\boldsymbol{u}} = 0$. Thus, the three-component spin is a tetrad (pseudo)vector. Such a definition of the three-component spin was used in Refs. [5, 25].

The definition of the three-component spin as a tetrad (pseudo)vector can be additionally justified by its consistency with the definition of the spin operator in quantum theory. The covariant Dirac equation for spin-1/2 particles in curved spacetimes has the form

$$(i\hbar\gamma^a D_a - mc)\psi = 0\,, \qquad a = 0, 1, 2, 3, \tag{29}$$

where γ^a are the Dirac matrices. The spinor covariant derivatives are defined by

$$D_a = e_a^\mu D_\mu\,, \qquad D_\mu = \partial_\mu + \frac{i}{4}\sigma_{ab}\Gamma_\mu^{ab}\,, \tag{30}$$

where $\Gamma_\mu^{ab} = -\Gamma_\mu^{ba}$ are the Lorentz connection coefficients, $\sigma^{ab} = i(\gamma^a\gamma^b - \gamma^b\gamma^a)/2$ (see Refs. [29,30] and references therein). Because the matrices γ^a are defined in the tetrad frame, they coincide with the Dirac matrices.

To obtain the equations of motion of particles and their spins, one can in principle use any tetrad. However, it does not mean that a choice of the tetrad is not important. In Eq. (24), the angular velocity of spin rotation should correspond to the quantity measured by a local observer. As a result, parameters of the definite tetrad frame carried by the observer should be substituted into this equation. For the observer in a uniformly accelerated, rotating frame, the tetrad $\boldsymbol{e}_a$ transports along the observer's world line according to [28]

$$\frac{d\boldsymbol{e}_a}{d\tau} = \boldsymbol{\Xi}\boldsymbol{e}_a\,, \tag{31}$$

where $\boldsymbol{\Xi}$ is the antisymmetric rotation tensor. This tensor consists of the Fermi–Walker part $\boldsymbol{\Xi}_{\rm FW}$ and the spatial rotation part $\boldsymbol{\Xi}_{\rm R}$ [28]:

$$\Xi^{\mu\nu} = \Xi^{\mu\nu}_{\rm FW} + \Xi^{\mu\nu}_{\rm R}\,, \qquad \Xi^{\mu\nu}_{\rm FW} = a^\mu u^\nu - a^\nu u^\mu\,, \qquad \Xi^{\mu\nu}_{\rm R} = u_\alpha \omega_\beta \epsilon^{\alpha\beta\mu\nu}\,, \tag{32}$$

where a^μ is the four-acceleration of the observer, ω^μ is its four-rotation, and $\epsilon^{\alpha\beta\mu\nu}$ is the Levi–Civita tensor.

For the uniformly accelerated, rotating frame, the exact formula for the orthonormal tetrad satisfying Eqs. (31), (32) was found by Hehl and Ni [31]. The corresponding vierbein has the form

$$e_0^{\hat{0}} = 1 + \frac{\boldsymbol{a}\cdot\boldsymbol{r}}{c^2}\,, \qquad e_i^{\hat{0}} = 0\,, \qquad e_0^{\hat{i}} = -g_{0i} = \frac{(\boldsymbol{\omega}\times\boldsymbol{r})^i}{c}\,, \qquad e_j^{\hat{i}} = \delta_{ij}\,, \tag{33}$$

where δ_{ij} is the Kronecker delta. Vierbein (33) is attributed to the observer being at rest in the uniformly accelerated, rotating frame [28,31].

Since the equivalence principle predicts the equivalence of gravitational fields and noninertial frames, the result obtained in Ref. [31] can be used for any spacetime defined by a metric tensor with $g_{0i} \neq 0$. Nonzero g_{0i} components are connected with the proper local three-rotation $\boldsymbol{\omega}$. In the weak-field approximation, the generalization of the tetrad found in Ref. [31] is given by

$$e_0^{\hat{0}} = 1 + \tfrac{1}{2}\,h_{00}\,, \qquad e_i^{\hat{0}} = 0\,, \qquad e_0^{\hat{i}} = -g_{0i}\,, \qquad e_j^{\hat{i}} = \delta_{ij} - \tfrac{1}{2}\,h_{ij}\,. \tag{34}$$

This vierbein can also be presented in the equivalent forms

$$e_{\hat{0}0} = 1 + \tfrac{1}{2}\,h_{00}\,, \qquad e_{\hat{0}i} = 0\,, \qquad e_{\hat{i}0} = g_{0i}\,, \qquad e_{\hat{i}j} = -\delta_{ij} + \tfrac{1}{2}\,h_{ij} \tag{35}$$

and

$$e^0_{\hat{0}} = 1 - \tfrac{1}{2}\,h_{00}\,, \qquad e^i_{\hat{0}} = g_{0i}\,, \qquad e^0_{\hat{i}} = 0\,, \qquad e^j_{\hat{i}} = \delta_{ij} + \tfrac{1}{2}\,h_{ij}\,. \tag{36}$$

Vierbeins (34)–(36) are connected with the observer at rest.

The nonzero Ricci rotation coefficients are equal to

$$\begin{aligned}\gamma_{i00} &= \tfrac{1}{2}\, g_{00,i} = -\gamma_{0i0}\,, \qquad \gamma_{i0j} = \tfrac{1}{2}\,(g_{0i,j} + g_{0j,i}) = -\gamma_{0ij}\,,\\ \gamma_{ij0} &= \tfrac{1}{2}\,(g_{0j,i} - g_{0i,j})\,, \qquad \gamma_{ijk} = \tfrac{1}{2}\,(g_{jk,i} - g_{ik,j})\,.\end{aligned} \tag{37}$$

Alternatively, one can use the vierbeins proposed by Pomeransky and Khriplovich [5]

$$e^{\hat{0}}_0 = 1 + \tfrac{1}{2}\, h_{00}\,, \qquad e^{\hat{0}}_i = \tfrac{1}{2}\, g_{0i}\,, \qquad e^{\hat{i}}_0 = -\tfrac{1}{2}\, g_{0i}\,, \qquad e^{\hat{i}}_j = \delta_{ij} - \tfrac{1}{2}\, h_{ij}\,, \tag{38}$$

and Landau and Lifshitz [23]

$$e^{\hat{0}}_0 = 1 + \tfrac{1}{2}\, h_{00}\,, \qquad e^{\hat{0}}_i = g_{0i}\,, \qquad e^{\hat{i}}_0 = 0\,, \qquad e^{\hat{i}}_j = \delta_{ij} - \tfrac{1}{2}\, h_{ij}\,. \tag{39}$$

Formulae (34)–(39) are given in the weak-field approximation.

The expression for the Ricci rotation coefficients obtained in Ref. [5] with the Pomeransky–Khriplovich tetrad differs from Eq. (37):

$$\gamma_{abc} = \tfrac{1}{2}\,(h_{bc,a} - h_{ac,b}) = \tfrac{1}{2}\,(g_{bc,a} - g_{ac,b})\,. \tag{40}$$

If a tetrad does not satisfy Eqs. (31), (32), it is not attributed to the observer's frame. We can consider an influence of the choice of the tetrad on equation of spin motion (24). The connection between different tetrad frames can be expressed by appropriate Lorentz transformations. Let the vierbeins e^μ_a and e'^μ_a define two tetrad frames and the unprimed vierbein is attributed to the observer's rest frame. The connection between tetrad and world coordinates is given by

$$dx^a = e^a_\mu dx^\mu\,, \qquad dx'^a = e'^a_\mu dx^\mu\,. \tag{41}$$

Eq. (41) leads to the relationship between tetrad coordinates in two frames:

$$dx^a = T^a_b dx'^b\,, \qquad T^a_b = e^a_\mu e'^\mu_b\,. \tag{42}$$

Since this relationship defines a Lorentz transformation, the primed frame moves with the relative velocity $\boldsymbol{V}$. This velocity is equal to zero only when $T^{\hat{0}}_{\hat{i}} = 0,\ T^{\hat{i}}_{\hat{0}} = 0,\ i = 1,2,3.$ In this case, the primed frame can be obtained from the unprimed one with a turn of the triad $\boldsymbol{e}_{\hat{i}}$ in the three-dimensional space. Evidently, this turn does not change the observer's rest frame and the condition $\boldsymbol{V} = 0$ remains valid. Such a turn does not distort the dynamics of particles and their spins, while it changes the connection between world and tetrad velocities.

In the general case, the primed tetrad is attributed to the reference frame moving with the velocity $\boldsymbol{V}$ relative to the observer. As a rule, this velocity is time-dependent. Eq. (28) shows that the three-component spins $\boldsymbol{S}$ and $\boldsymbol{S}'$ are defined in different reference frames. Since accelerations of these frames do not equal, the Thomas precession causes a difference between angular velocities of rotation of the (pseudo)vectors $\boldsymbol{S}$ and $\boldsymbol{S}'$.

As a result, the reason of the change in the spin motion equation is the Thomas precession. For relativistic particles, the angular velocity of the Thomas precession is given by [15, 32]

$$\boldsymbol{\Omega}_{\rm T} = -\frac{\gamma}{\gamma+1}(\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}})\,, \tag{43}$$

where $\gamma = 1/\sqrt{1-\beta^2}$ is the Lorentz factor.

The dependence of spin motion equation (24) from the choice of the tetrad was not taken into account in Refs. [5, 25]. The tetrad used in Refs. [5, 25] for a derivation of equations of spin motion in the world frame satisfies Eqs. (31), (32) only for static spacetimes. To determine the observable angular velocity for nonstatic spacetimes, one needs to supplement the PKE with the correction for the Thomas precession.

To illustrate a dependence of Eq. (24) from the choice of the tetrad, we consider the spin motion in the rotating frame. This problem can be solved exactly. The metric tensor is given by [31]

$$g_{\mu\nu} = \begin{pmatrix} 1-\frac{(\boldsymbol{\omega}\times\boldsymbol{r})^2}{c^2} & -\frac{(\boldsymbol{\omega}\times\boldsymbol{r})^{(1)}}{c} & -\frac{(\boldsymbol{\omega}\times\boldsymbol{r})^{(2)}}{c} & -\frac{(\boldsymbol{\omega}\times\boldsymbol{r})^{(3)}}{c} \\ -\frac{(\boldsymbol{\omega}\times\boldsymbol{r})^{(1)}}{c} & -1 & 0 & 0 \\ -\frac{(\boldsymbol{\omega}\times\boldsymbol{r})^{(2)}}{c} & 0 & -1 & 0 \\ -\frac{(\boldsymbol{\omega}\times\boldsymbol{r})^{(3)}}{c} & 0 & 0 & -1 \end{pmatrix}, \tag{44}$$

where $\boldsymbol{\omega}$ is the angular frequency of the frame rotation. The use of Eqs. (26), (34)–(36) results in the following equation of particle motion:

$$\frac{d\hat{\boldsymbol{u}}}{dt} = -\boldsymbol{\omega}\times\hat{\boldsymbol{u}}\,, \qquad \frac{du^{\hat{0}}}{dt} = 0\,. \tag{45}$$

Eq. (45) leads to the right equation for the contravariant acceleration $du^\mu/(dt)$ coinciding with the well-known formula [33] for the sum of the Coriolis and centrifugal accelerations.

The corresponding angular velocity of spin motion obtained from Eq. (24) is given by

$$\boldsymbol{\Omega} = -\boldsymbol{\omega}\,. \tag{46}$$

This formula is also exact and coincides with previous results [31, 34, 35].

Pomeransky–Khriplovich vierbein (38) leads to the formula

$$\boldsymbol{\Omega} = -\boldsymbol{\omega} + \frac{\boldsymbol{u} \times (\boldsymbol{u} \times \boldsymbol{\omega})}{2u^0(u^0+1)}\,. \tag{47}$$

The use of Landau–Lifshitz vierbein (39) results in

$$\boldsymbol{\Omega} = -\boldsymbol{\omega} + \frac{\boldsymbol{u} \times (\boldsymbol{u} \times \boldsymbol{\omega})}{u^0(u^0+1)}\,. \tag{48}$$

Eqs. (47), (48) are obtained in the weak-field approximation. Evidently, these equations do not give the observable angular velocity defined by Eqs. (46).

6. Comparison of spin effects in classical and quantum gravity

The correspondence principle formulated by Niels Bohr predicts a similarity of classical and quantum effects.

The best compliance between the description of spin effects in classical and quantum gravity was proved in Refs. [36, 37]. In these works, some Hamiltonians in the Dirac representation derived in Refs. [29–31] from initial Dirac equation (29) were used. The initial Dirac Hamiltonians were transformed to the Foldy–Wouthyusen (FW) representation by the method elaborated in Ref. [38]. The FW representation [39] occupies a special place in the quantum theory. Properties of this representation are unique. The Hamiltonian and all operators are block-diagonal (diagonal in two spinors). Relations between the operators in the FW representation are similar to those between the respective classical quantities. For relativistic particles in external fields, operators have the same form as in the nonrelativistic quantum theory. For example, the position operator is $\boldsymbol{r}$ and the momentum one is $\boldsymbol{p} = -i\hbar\nabla$. These properties considerably simplify the transition to the semiclassical description. As a result, the FW representation provides the best possibility of obtaining a meaningful classical limit of the relativistic quantum mechanics. The basic advantages of the FW representation are described in Refs. [38–40]. The method of the FW transformation for relativistic particles in external fields was proposed in Ref. [38].

The exact transformation of the Dirac equation for the metric

$$ds^2 = V^2(\boldsymbol{r})(dx^0)^2 - W^2(\boldsymbol{r})(d\boldsymbol{r}\cdot\boldsymbol{r}) \tag{49}$$

to the Hamiltonian form was carried out by Obukhov [29, 30] ($\hbar = c = 1$):

$$i\frac{\partial\psi}{\partial t} = \mathcal{H}\psi\,, \qquad \mathcal{H} = \beta m V + \tfrac{1}{2}\left\{\mathcal{F}, \boldsymbol{\alpha}\cdot\boldsymbol{p}\right\}, \tag{50}$$

where $\mathcal{F}=V/W$. Hamiltonian (50) covers the cases of a weak Schwarzschild field in the isotropic coordinates

$$V=\left(1-\frac{GM}{2r}\right)\left(1+\frac{GM}{2r}\right)^{-1}, \qquad W=\left(1+\frac{GM}{2r}\right)^{2}, \tag{51}$$

and a uniformly accelerated frame

$$V=1+\boldsymbol{a}\cdot\boldsymbol{r}, \qquad W=1. \tag{52}$$

The relativistic FW Hamiltonian derived in Ref. [36] has the form

$$\begin{aligned}\mathcal{H}_{\mathrm{FW}} \;=\; & \beta\epsilon+\frac{\beta}{2}\left\{\frac{m^2}{\epsilon},V-1\right\}+\frac{\beta}{2}\left\{\frac{\boldsymbol{p}^2}{\epsilon},\mathcal{F}-1\right\} \\ & -\frac{\beta m}{4\epsilon(\epsilon+m)}\left[\boldsymbol{\Sigma}\cdot(\boldsymbol{\phi}\times\boldsymbol{p})-\boldsymbol{\Sigma}\cdot(\boldsymbol{p}\times\boldsymbol{\phi})+\nabla\cdot\boldsymbol{\phi}\right] \\ & +\frac{\beta m(2\epsilon^3+2\epsilon^2m+2\epsilon m^2+m^3)}{8\epsilon^5(\epsilon+m)^2}\left(\boldsymbol{p}\cdot\nabla\right)\left(\boldsymbol{p}\cdot\boldsymbol{\phi}\right) \\ & +\frac{\beta}{4\epsilon}\left[\boldsymbol{\Sigma}\cdot(\boldsymbol{f}\times\boldsymbol{p})-\boldsymbol{\Sigma}\cdot(\boldsymbol{p}\times\boldsymbol{f})+\nabla\cdot\boldsymbol{f}\right]-\frac{\beta(\epsilon^2+m^2)}{4\epsilon^5}(\boldsymbol{p}\cdot\nabla)(\boldsymbol{p}\cdot\boldsymbol{f}),\end{aligned} \tag{53}$$

where $\epsilon=\sqrt{m^2+\boldsymbol{p}^2}$, $\boldsymbol{\phi}=\nabla V$, $\boldsymbol{f}=\nabla\mathcal{F}$.

The use of the FW representation dramatically simplifies the derivation of quantum equations. The operator equations of momentum and spin motion obtained via commutators of the Hamiltonian with the momentum and polarization operators take the form [36]

$$\begin{aligned}\frac{d\boldsymbol{p}}{dt} \;=\; & i\left[\mathcal{H}_{\mathrm{FW}},\boldsymbol{p}\right]=-\frac{\beta}{2}\left\{\frac{m^2}{\epsilon},\boldsymbol{\phi}\right\}-\frac{\beta}{2}\left\{\frac{\boldsymbol{p}^2}{\epsilon},\boldsymbol{f}\right\} \\ & +\frac{m}{2\epsilon(\epsilon+m)}\nabla\bigl(\boldsymbol{\Pi}\cdot(\boldsymbol{\phi}\times\boldsymbol{p})\bigr)-\frac{1}{2\epsilon}\nabla\bigl(\boldsymbol{\Pi}\cdot(\boldsymbol{f}\times\boldsymbol{p})\bigr)\end{aligned} \tag{54}$$

and

$$\frac{d\boldsymbol{\Pi}}{dt} \;=\; i[\mathcal{H}_{\mathrm{FW}},\boldsymbol{\Pi}]=\frac{m}{\epsilon(\epsilon+m)}\boldsymbol{\Sigma}\times(\boldsymbol{\phi}\times\boldsymbol{p})-\frac{1}{\epsilon}\boldsymbol{\Sigma}\times(\boldsymbol{f}\times\boldsymbol{p}), \tag{55}$$

respectively.

Let us pass to the studies of semiclassical limit of these equations. The contribution of the lower spinor is negligible and the transition to the semiclassical description is performed by averaging the operators in the equations for the upper spinor [38]. It is usually possible to neglect the commutators between the coordinate and momentum operators. As a result, the operators $\boldsymbol{\sigma}$ and $\boldsymbol{p}$ should be substituted by the corresponding classical quantities: the polarization vector (doubled average spin), $\boldsymbol{\xi}$, and the momentum. For the latter quantity, we retain the notation $\boldsymbol{p}$. The semiclassical equations of motion are [36]

$$\frac{d\boldsymbol{p}}{dt}=-\frac{m^2}{\epsilon}\boldsymbol{\phi}-\frac{\boldsymbol{p}^2}{\epsilon}\boldsymbol{f}+\frac{m}{2\epsilon(\epsilon+m)}\nabla\big(\boldsymbol{\xi}\cdot(\boldsymbol{\phi}\times\boldsymbol{p})\big)-\frac{1}{2\epsilon}\nabla\big(\boldsymbol{\xi}\cdot(\boldsymbol{f}\times\boldsymbol{p})\big) \quad (56)$$

and

$$\frac{d\boldsymbol{\xi}}{dt}=\frac{m}{\epsilon(\epsilon+m)}\boldsymbol{\xi}\times(\boldsymbol{\phi}\times\boldsymbol{p})-\frac{1}{\epsilon}\boldsymbol{\xi}\times(\boldsymbol{f}\times\boldsymbol{p})\ , \quad (57)$$

respectively. In Eq. (56), two latter terms describe a force dependent on the spin. This force is similar to the electromagnetic Stern–Gerlach force (see Ref. [38]). Because it is weak, the approximate semiclassical equation of particle motion takes the form

$$\frac{d\boldsymbol{p}}{dt}=-\frac{m^2}{\epsilon}\boldsymbol{\phi}-\frac{\boldsymbol{p}^2}{\epsilon}\boldsymbol{f}\,. \quad (58)$$

The angular velocity of spin rotation is given by

$$\boldsymbol{\Omega}=-\frac{m}{\epsilon(\epsilon+m)}\left(\boldsymbol{\phi}\times\boldsymbol{p}\right)+\frac{1}{\epsilon}\left(\boldsymbol{f}\times\boldsymbol{p}\right)\,. \quad (59)$$

We can find similar equations describing a change of the direction of particle momentum, $\boldsymbol{n}=\boldsymbol{p}/p$:

$$\frac{d\boldsymbol{n}}{dt}=\boldsymbol{\omega}\times\boldsymbol{n}\,,\qquad \boldsymbol{\omega}=\frac{m^2}{\epsilon p}\big(\boldsymbol{\phi}\times\boldsymbol{n}\big)+\frac{p}{\epsilon}\big(\boldsymbol{f}\times\boldsymbol{n}\big)\,. \quad (60)$$

A simple calculation shows that the corresponding equations of motion obtained from the PKE for given metric (49) coincide with Eqs. (56)–(60). In particular, the gravitational Stern–Gerlach force defined by Eq. (56) coincides with that obtained from the PKE. The comparison with previous results obtained in framework of classical gravity was carried out in Ref. [36].

Although the gravitational Stern–Gerlach forces are rather weak, they are important. These forces lead to the violation of the weak equivalence principle due to deflections of spinning particles from geodesic lines [41].

Let us consider the interaction of particles with a spherically symmetric gravitational field and compare the obtained formulae with previous results. This field is a weak limit of the Schwarzschild one which yields

$$V=1-\frac{GM}{r}\,,\qquad W=1+\frac{GM}{r}\,. \quad (61)$$

When we neglect the terms of the order of $\frac{(\boldsymbol{p}\cdot\nabla)(\boldsymbol{p}\cdot\boldsymbol{g})}{\epsilon^2}$, Hamiltonian (54) takes the form

$$\mathcal{H}_{\rm FW}=\beta\epsilon-\frac{\beta}{2}\left\{\frac{\epsilon^2+\boldsymbol{p}^2}{\epsilon},\frac{GM}{r}\right\}-\frac{\beta(2\epsilon+m)}{4\epsilon(\epsilon+m)}\Bigl[2\boldsymbol{\Sigma}\cdot(\boldsymbol{g}\times\boldsymbol{p})+\nabla\cdot\boldsymbol{g}\Bigr], \quad (62)$$

where $\boldsymbol{g}$ is the Newtonian acceleration.

Neglecting the Stern–Gerlach force, one gets the semiclassical expressions for the angular velocities of rotation of unit momentum vector, $\boldsymbol{n}=\boldsymbol{p}/p$, and spin:

$$\boldsymbol{\omega} \;=\; -\frac{\epsilon^2+\boldsymbol{p}^2}{\epsilon\boldsymbol{p}^2}\,\boldsymbol{g}\times\boldsymbol{p}=\frac{GM}{r^3}\cdot\frac{\epsilon^2+\boldsymbol{p}^2}{\epsilon\boldsymbol{p}^2}\,\boldsymbol{l}\,, \quad (63)$$

$$\boldsymbol{\Omega} \;=\; -\frac{2\epsilon+m}{\epsilon(\epsilon+m)}\,\boldsymbol{g}\times\boldsymbol{p}=\frac{GM}{r^3}\cdot\frac{2\epsilon+m}{\epsilon(\epsilon+m)}\,\boldsymbol{l}\,, \quad (64)$$

where $\boldsymbol{l}=\boldsymbol{r}\times\boldsymbol{p}$ is the angular moment.

Eqs. (63) and (64) agree with the classical gravity. Eq. (63) leads to the expression for the angle of particle deflection by a gravitational field

$$\theta=\frac{2GM}{\rho}\left(2+\frac{m^2}{\boldsymbol{p}^2}\right)=\frac{2GM}{\rho\boldsymbol{v}^2}\left(1+\boldsymbol{v}^2\right) \quad (65)$$

coinciding with Eq. (13) of Problem 15.9 from Ref. [42] (see also Ref. [43]). Eq. (64) coincides with the corresponding classical equation obtained in Ref. [5]. This directly proves the full compatibility of quantum and classical considerations.

In the nonrelativistic approximation, Eq. (64) coincides with corresponding formula (8) for the classical gyroscope. Such a similarity [13] of classical and quantum rotators is a manifestation of the equivalence principle (see *e.g.* Refs. [14, 44] and references therein). In the nonrelativistic approximation, the last term in Hamiltonian (62) describing the spin-orbit and contact (Darwin) interactions coincides with the corresponding term in Ref. [45].

Performing the FW transformation for relativistic particles made it possible to solve the problem of existence of the dipole spin-gravity coupling in a static gravitational field [36]. This problem was discussed for a long time (see Refs. [12, 29, 30, 36] and references therein). Evidently, this coupling given in form (9) contradicts to the theory [5, 36] and violates both the CP invariance and the relation predicting the absence of the GDM [13]. The classical and quantum approaches lead to the same conclusion.

The equation for the Hamiltonian and the equations of momentum and spin motion derived in Ref. [36] for a relativistic particle in a uniformly accelerated frame agree with the corresponding nonrelativistic expressions from [31, 46]. The general equations for the angular velocities of rotation of unit momentum vector and spin are given by

$$\boldsymbol{\omega} = \frac{\epsilon}{\boldsymbol{p}^2}(\boldsymbol{a}\times\boldsymbol{p}), \qquad \boldsymbol{\Omega} = \frac{\boldsymbol{a}\times\boldsymbol{p}}{\epsilon+m}. \tag{66}$$

The FW Hamiltonian and the operators of velocity and acceleration were also calculated for the Dirac particle in the rotating frame [37]. The exact Dirac Hamiltonian derived in Ref. [31] was used. In Ref. [37], perfect agreement between classical and quantum approaches was also established. The operators of velocity and acceleration are equal to

$$\begin{aligned} \boldsymbol{v} &= \beta\frac{\boldsymbol{p}}{\epsilon} - \boldsymbol{\omega}\times\boldsymbol{r}, \qquad \epsilon = \sqrt{m^2+\boldsymbol{p}^2}, \\ \boldsymbol{w} &= 2\beta\frac{\boldsymbol{p}\times\boldsymbol{\omega}}{\epsilon} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}) = 2\boldsymbol{v}\times\boldsymbol{\omega} - \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}). \end{aligned} \tag{67}$$

Quantum formula (67) for the acceleration of the relativistic spin-1/2 particle coincides with the classical formula [33] for the sum of the Coriolis and centrifugal accelerations. Obtained results also agree with the corresponding nonrelatiistic formulae from [31].

Thus, the classical and quantum approaches are in full agreement. Purely quantum effects are not too important. They consist in appearing some additional terms in the FW Hamiltonian. However, these terms are proportional to derivatives of $\boldsymbol{\phi}$ and $\boldsymbol{f}$ and similar to the well-known Darwin term in the electrodynamics. As a result, their influence on the motion of particles and their spins in gravitational fields can be neglected. The classical and quantum equations of motion of particles and their spins are almost identical and can differ only in small terms.

7. Equivalence principle and spin

As mentioned above, the absence of the AGM and GDM is very similar to the weak equivalence principle. All classical and quantum spins (gyroscopes) precess with the same angular velocity, while all classical and quantum particles move with the same acceleration. An equivalence of the inertia and gravity manifests in the fact that all gravitational and inertial phenomena are exhaustively defined by the metric tensor and four-velocity.

Another manifestation of the equivalence principle was found in Ref. [36]. It was shown that the motion of momentum and spin differs in a static gravitational field and a uniformly accelerated frame but the helicity evolution coincides. In Eqs. (59), (60) $\boldsymbol{\phi}$ depends only on g_{00} but $\boldsymbol{f}$ is also a function of g_{ij}.

The spin rotates with respect to the momentum direction and the angular velocity of this rotation is

$$\boldsymbol{o}=\boldsymbol{\Omega}-\boldsymbol{\omega}=-\frac{m}{p}\big(\boldsymbol{\phi}\times\boldsymbol{n}\big). \tag{68}$$

The quantity $\boldsymbol{o}$ does not depend on $\boldsymbol{f}$ and vanishes for massless particles. Therefore, the gravitational field cannot change the helicity of massless Dirac particles. The evolution of the helicity $\zeta\equiv|\boldsymbol{\xi}_\parallel|=\boldsymbol{\xi}\cdot\boldsymbol{n}$ of massive particles is defined by the formula

$$\frac{d\zeta}{dt}=(\boldsymbol{\Omega}-\boldsymbol{\omega})\cdot(\boldsymbol{\xi}_\perp\times\boldsymbol{n})=-\frac{m}{p}\left(\boldsymbol{\xi}_\perp\cdot\boldsymbol{\phi}\right), \tag{69}$$

where $\boldsymbol{\xi}_\perp=\boldsymbol{\xi}-\boldsymbol{\xi}_\parallel$.

The same formulae can be derived from the PKE.

For particles in the spherically symmetric gravitational field, formula (68) takes the form

$$\boldsymbol{o}=\boldsymbol{\Omega}-\boldsymbol{\omega}=\frac{m}{\boldsymbol{p}^2}\big(\boldsymbol{g}\times\boldsymbol{p}\big)\,. \tag{70}$$

If the angle of particle momentum deflection θ is small, the evolution of the helicity is described by the equation [36]

$$\zeta=1-\frac{\theta^2}{2(2\gamma-\gamma^{-1})^2}\,, \tag{71}$$

where $\gamma=\epsilon/m$ is the Lorentz factor. The original helicity is supposed to be $+1$.

The relative angular velocity defining the helicity evolution in the uniformly accelerated frame is given by

$$\boldsymbol{o}=\boldsymbol{\Omega}-\boldsymbol{\omega}=-\frac{m}{\boldsymbol{p}^2}\big(\boldsymbol{a}\times\boldsymbol{p}\big)\,. \tag{72}$$

When $\boldsymbol{a}=-\boldsymbol{g}$, values of $\boldsymbol{o}$ in Eqs. (72) and (70) are the same. It is the manifestation of the equivalence principle which was discussed with respect to helicity evolution in [14, 44].

At the same time, the manifestation of the equivalence principle for the spin rotation is not so trivial. In particular, the spin of nonrelativistic particles in the spherically symmetric gravitational field rotates three times more rapidly in comparison with the corresponding accelerated frame [36].

The helicity evolution caused by the rotation of an astrophysical object was considered in Ref. [14]. The effect of the rotation of a field source is characterized by the gravitomagnetic field. This field makes the velocity rotate twice faster than the spin and changes the helicity. Therefore, the helicity can locally evolve due to the rotation of the field source. Nevertheless, the integral effect for the particle passing throughout the gravitational field region is zero. Thus, the helicity of the scattered massive particle is not influenced by the rotation of an astrophysical object [14]. Some other authors came to the alternative conclusion that the above mentioned rotation changes the helicity of the scattered massive particle [47,48]. To obtain a definite solution of this problem, the PKE and the Dirac equation can be used.

8. Discussion and summary

The general equations describing the dynamics of classical spin in gravitational fields and noninertial frames was obtained by Mathisson and Papapetrou [4, 6] and by Pomeransky and Khriplovich [5]. The MPE and PKE are different in principle. Nevertheless, the spin dynamics predicted by the MPE and PKE is the same in the first-order approximation in the spin. This important conclusion shows that the Mathisson–Papapetrou and Pomeransky–Khriplovich approaches lead to the same observable spin effects. Results obtained with two approaches differ only in terms of the second and higher orders in spin. These terms are proportional to derivatives of the second and higher orders of the metric tensor. Both of approaches predict the violation of the weak equivalence principle due to deflections of spinning particles from geodesic lines. Such deflections are caused by the gravitational Stern–Gerlach forces which are rather weak. Nevertheless, these forces are important because they condition the violation of the weak equivalence principle [41]. The Mathisson–Papapetrou and Pomeransky–Khriplovich approaches give different expressions for the gravitational Stern–Gerlach forces. The expression resulting from the PKE agrees with that obtained from the Dirac equation.

The PKE are rather convenient for description of spin motion in the framework of classical gravity. The general equation of spin motion [5] is valid in arbitrary spacetimes. However, the angular velocity of spin precession defined by Eq. (24) depends on the choice of the tetrad. The origin of such a dependence is the fact that reference frames defined by different tetrads can move relatively to each other. In this case, the corresponding angular velocities of spin precession differ due to the Thomas precession. We derive the exact equation describing the spin dynamics in the rotating frame.

An important property of spin interactions with curved spacetimes is the absence of the AGM and GDM [13]. The relations obtained by Kobzarev and Okun lead to equal frequencies of precession of classical and quantum spins in curved spacetimes and the preservation of helicity of Dirac particles in gravitomagnetic fields [14]. As a result, the behavior of classical and quantum spins in curved spacetimes is the same and any quantum effects cannot appear. However, this point of view was not generally accepted until very recently.

The fact that dynamics of classical and quantum spins in curved spacetimes is identical was also proved in Refs. [36, 37]. The full agreement between classical equations of momentum and spin motion and corresponding quantum equations obtained from solution of the Dirac equation was established. The classical and quantum equations was compared not only for gravitational fields but also for noninertial frames. The absence of any fundamentally new spin effects is a manifestation of the correspondence principle.

Another manifestation of the equivalence principle is the helicity evolution. While the motion of momentum and spin differs in static gravitational fields and uniformly accelerated frames, the helicity evolution is the same [36].

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