LORENTZ COVARIANCE OF LANGEVIN EQUATION*

T. Koide, G.S. Denicol, T. Kodama

Instituto de Física, Universidade Federal do Rio de Janeiro
C.P. 68528, 21945-970, Rio de Janeiro, Brazil

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Relativistic covariance of a Langevin type equation is discussed. The requirement of Lorentz invariance generates an entanglement between the force and noise terms so that the noise itself should not be a covariant quantity.

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1. Introduction

When a stochastic dynamics is involved in a relativistic process, it is not trivial to formulate the problem in a Lorentz invariant way. A typical example is a cascade calculation for a many particle system in a relativistic energy, such as relativistic intra-nuclear cascade or parton cascade models. These calculations are not covariant and the difference in the time-ordering of collisions in cascade calculation even affects the equilibrium distribution in momentum space as shown below.

Consider thermal equilibration among N cascading relativistic particles in a box of volume V. To proceed the cascade calculation, we first establish the timetable of every possible binariy collisions, then perform the collisions one by one according to the timetable. After each collision, the momenta and the timetable should be up-dated [1]. In the homogeneous case where we are only interested in the momentum distribution, one might think that the cascade process can be replaced by choosing the binary collision randomly. This is equivalent to the assumption of molecular chaos in the case of Boltzmann equation, so one expects the resulting single particle distribution to be that of Boltzmann, $dN/dp^3 \propto e^{-\beta E}$.

After a large enough number of binary collisions, it is seen that the single particle spectrum in fact converges, but to $e^{-\beta E}/E$ rather than the Boltzmann distribution, as seen in Fig. 1(a). This is somewhat an expected

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result, since in this calculation, collision events are assumed to occur homogeneously in momentum space without information of the space distribution of the particles so what we obtain will be the same as the single particle distribution in invariant phase volume calculation [2]. There, the normalization of the distribution function is defined by $\int d^3p\, f(\vec{p}) = \mathrm{const.}$ In this case, since d^3p/E is a scalar measure, $Ef(p)$ should be a Lorentz scalar.

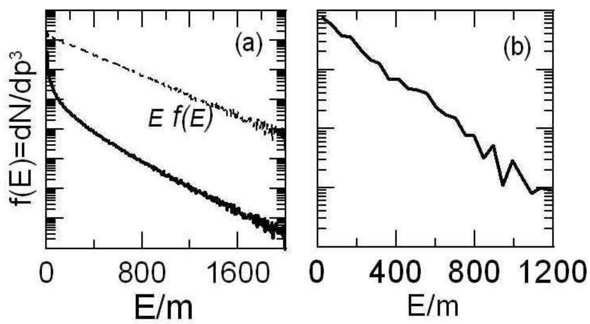

Fig. 1. Single particle spectra of relativistic cascade calculation. (a) The results of random collisions in the momentum space. The upper line corresponds to $Ef(E)$. (b) The result of space-time cascade calculation, showing the Boltzmann distribution.

On the other hand, if we really perform the binary collisions following all the particle trajectories using the covariant impact parameter method [1], the single particle spectrum recovers the usual Boltzmann distribution as seen in Fig. 1(b). Here the single particle distribution is normalized together with the space part as $\int d^3\vec{r} \int d^3\vec{p}\, f(\vec{r}, \vec{p}) = \mathrm{const.}$ so that $Vf(\vec{p})$ is a scalar, where V is the volume of the system. The above example shows a quite interesting feature of cascade type calculations with respect to the Lorentz covariance and the single particle spectra. It is worthwhile to investigate more precisely the time evolution and Lorentz covariance of single particle distribution in such a system. For non-relativistic cases, the Langevin equation for a Brownian motion is a useful approach to investigate the dynamics of single particle distribution of stochastic processes. In this work we discuss the relativistic generalization of the Brownian motion [4].

2. Lorentz covariance of relativistic Brownian motion

We consider the Brownian motion of a relativistic particle with mass m in the 3+1 dimension described by the stochastic differential equation (SDE) defined on the discretized time sets, $\{t_i^* = i\ dt^*, i = 0, 1, \ldots\}$

$$d\boldsymbol{x}^* = \frac{\boldsymbol{p}^*}{p^{0*}} dt^* , \qquad d\boldsymbol{p}^* = -\nu(p^{0*})\boldsymbol{p}^* dt^* + \sqrt{2D(p^{0*})}\, d\boldsymbol{w}_{t^*} . \tag{1}$$

where the stochastic variable $d\boldsymbol{w}_{t^*}$ satisfies the usual properties of Gaussian noise, $\langle d\boldsymbol{w}^*_{t^*_i}\rangle_0 = 0$, $\langle d\boldsymbol{w}^{i*}_{t^*_k} d\boldsymbol{w}^{j*}_{t^*_l}\rangle_0 = dt^*\delta_{ij}\delta_{kl}$. Eq. (1) contains the multiplicative noise and the solution depends on the integration scheme of the stochastic term. Here we consider the three different schemes of Ito, Stratonovich–Fisk and Hänggi–Klimontovich [3].

From the SDE and the integration rules above, we can construct the corresponding Fokker–Planck equation and obtain the equilibrium form of the distribution function. It is given as

$$\rho_{\rm st}(\boldsymbol{x}^*, \boldsymbol{p}^*) \propto \exp\left(- \int_C^{\boldsymbol{p}^*} d\boldsymbol{q} \cdot \frac{\boldsymbol{q}\, \nu(q^{0*})}{D(q^{0*})} - \alpha \ln D\left(p^{0*}\right) \right). \tag{2}$$

Here, the parameter α takes the values $0, 1/2$ and 1, corresponding to the cases of Hänggi–Klimontovich, Stratonovich–Fisk and Ito scheme, respectively.

Now, let us consider the reference frame which is moving with the velocity $\boldsymbol{V}$ with respect to the rest frame of the heat bath (we refer to as simply *MF-moving frame*). The four-momentum dp^μ in this frame is then given by the Lorentz transformation of $dp^{*\mu}$. Using the on mass-shell condition $dp^0 = -\boldsymbol{p}\cdot d\boldsymbol{p}/p^0$, we get the SDE in the MF which contains the stochastic term. However, the stochastic term is defined only in the rest frame of the bath. We now assume that, even after the Lorentz boost, the stochastic part of the Brownian motion still preserves the property of the Gaussian white noise, which is defined by $\langle d\boldsymbol{w}_t\rangle_V = 0$ and $\langle d\boldsymbol{w}^i_{t_l} d\boldsymbol{w}^j_{t_m}\rangle_V = dt\delta_{ij}\delta_{lm}$. Here the symbol $\langle X\rangle_V$ denotes the stochastic average of X in the MF. Now, it is important to note that the statistical average of the noise term defined in the different reference frame does not necessarily vanish. For example, we may have $\langle d\boldsymbol{w}^*_{t^*}\rangle_V \neq 0$. The reason is that $d\mathbf{w}^*_{t^*}$ is non-local in the time t so that the Lorentz transformation entangles with the integration scheme. This implies that the force part and the stochastic part can be mixed in the order of dt. Thus we have

$$d\boldsymbol{w}^*_{t^*} = \gamma^{1/2}(V)\sqrt{\frac{p^0 - \beta(V)p_V}{p^0}}\, d\boldsymbol{w}_t + \boldsymbol{C}_{\boldsymbol{p}}\, dt\,. \tag{3}$$

Here the non-vanishing average, $\boldsymbol{C}_{\boldsymbol{p}}\, dt$ is separated from the pure stochastic part $d\boldsymbol{w}_t$. The corresponding Langevin equation is given by

$$\begin{aligned} d\boldsymbol{p}^i &= -\frac{\nu(u^\mu p_\mu)}{p^0}\left\{p^0(\boldsymbol{p}^i - \beta(V)\boldsymbol{n}^i p^0) + \beta(V)(p^2\boldsymbol{n}^i - p_V\boldsymbol{p}^i)\right\}\, dt \\ &\quad + (1-\alpha)\sum_{jk}\tilde{\boldsymbol{B}}^{jk}\partial^j_{\boldsymbol{p}}\tilde{\boldsymbol{B}}^{ik} dt + [\boldsymbol{B}\boldsymbol{C}_p]^i dt + \left[\tilde{\boldsymbol{B}} d\boldsymbol{w}_t\right]^i, \end{aligned} \tag{4}$$

where $\tilde{\boldsymbol{B}} = \sqrt{\gamma(V)(p^0 - \beta(V)p_V)/p^0}\boldsymbol{B}$. Now we can derive the corresponding Fokker–Planck equation and its equilibrium distribution. We know that the distribution function is a Lorentz scalar, so that the equilibrium distribution should be equal to the one given in Eq. (2). From this, we find the vector $\boldsymbol{C_p}$ should satisfy [4]

$$\boldsymbol{C}^{\perp}_{\boldsymbol{p}} = -(1-\alpha)\sqrt{\frac{D(u^\mu p_\mu)}{2}}\frac{\beta(V)\gamma(V)}{(p^0)^2}\frac{p_V - \beta(V)p^0}{p^0 - \beta(V)p_V}\,, \tag{5}$$

$$\begin{aligned} \boldsymbol{C}^{\parallel}_{\boldsymbol{p}} = {} & \frac{\beta(V)}{(p^0)^2}\sqrt{\frac{D(u^\mu p_\mu)}{2}}\left[\frac{\alpha m^2}{p^0 - \beta(V)p_V}\right. \\ & + 2(1-2\alpha)^2\gamma(V)(p^0 - \beta(V)p_V) + (1-\alpha)\gamma(V) \\ & \left.\left\{\left(\frac{p^2}{p^0} - \beta(V)p_V\right) - \left(p^0 p_V - \beta(V)p^2\right)\frac{1}{p^0}\frac{p_V - \beta(V)p^0}{p^0 - \beta(V)p_V}\right\}\right], \end{aligned} \tag{6}$$

where we decompose $\boldsymbol{C_p}$ into the longitudinal and transversal components, $\boldsymbol{C_p} = \boldsymbol{C}^{\perp}_{\boldsymbol{p}}(\boldsymbol{p} - p_V\boldsymbol{n}) + \boldsymbol{C}^{\parallel}_{\boldsymbol{p}}\boldsymbol{n}$. We conclude that in order to keep the scalar property of the equilibrium distribution function, $\boldsymbol{C_p}$ cannot be null, which is a new result.

In this work, we discussed the generalization of the Brownian motion of a relativistic particle. The covariance of the SDE requires that the noise must be essentially multiplicative in a general frame. The Lorentz boost induces a non-trivial entanglement between the force term and the noise term. We demonstrated that the commonly used Lorentz invariant noise does not lead to an invariant equilibrium distribution in the present formulation of the relativistic Brownian motion.

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