THE EXTERNAL CLOCK AND THE DECAY OF A TWO-PARTICLE SYSTEM INSIDE A SPATIAL BOX*

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A series of experiments show that the physical time is the same kind of quantum observable as the spatial position. Using the projection evolution as the extension of the standard Schrödinger type evolution, the decay of a two-particle system in the case of limited allowed space (box) is considered. A very schematic model is used to show the size effects in the decay probability distribution.

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1. Introduction

In the standard approach to the quantum mechanics, time is considered as an events ordering parameter. The experiments in which the temporal interference is obtained show [1,2] that this concept is only an approximation and that the quantum time should be the same kind of a quantum observable as the position in space [3–5].

The time scale, in which the temporal quantum effects emerge, is of the order of femtoseconds and shorter. In order to treat the time as quantum observable, we developed the projection evolution formalism (PEv), see [7–12]. In this formalism, the quantum evolution is a stochastic process in

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which the subsequent quantum events are randomly created, with the probability distribution determined by the actual state of the system and the so-called evolution operators.

In this work, we show that the projection evolution approach allows to investigate temporal effects in the decay of nuclei and any other quantum systems in terms of the competition between remaining in the bound state and decaying. The key assumption here is that time is a dynamical variable, not a parameter. In this paper, we discuss the possible role of the external clock in the decay process. It is not our goal to provide a realistic scenario of these processes which, especially for fast processes as those observed in nuclear reactions, should include temporal effects, too. This paper is, to some extend, the continuation of [6].

2. Two-particle system

To schematically show a possible scenario of the decay process as the projection quantum evolution, we consider two spinless particles in the non-relativistic space-time $\boldsymbol{X}$. The relativistic case can be treated in an analogical way and will be described elsewhere.

First, we introduce the notation and conventions used in this paper. The coordinates of the n^{th} particle, $n = 1, 2, \ldots, A$, in the $N+1$ dimensional space-time are denoted by $x_n = (x_n^0, x_n^1, x_n^2, \ldots, x_n^N) = (ct_n, \vec{x}_n)$. Similarly, the linear momentum is denoted by $p_n = (p_{n0}, p_{n1}, p_{n2}, \ldots, p_{nN}) = (E_n/c, \vec{p}_n)$. The corresponding components of the linear momentum operator for the n^{th} particle are

$$(\hat{p}_{n0}, \hat{p}_{n1}, \hat{p}_{n2}, \ldots, \hat{p}_{nN}) = \left(i\hbar \frac{\partial}{\partial x_n^0}, -i\hbar \frac{\partial}{\partial x_n^1}, \ldots, -i\hbar \frac{\partial}{\partial x_n^N} \right) .$$

Next, we introduce the relative and the "center-of-mass" coordinates in the space-time $\boldsymbol{X}$. For a two-particle system ($A = 2$), they read

$$\begin{aligned} x^\mu &= x_2^\mu - x_1^\mu \,, \\ \xi^\mu &= \frac{1}{M_\mu} \left(m_{1\mu} x_1^\mu + m_{2\mu} x_2^\mu \right) , \end{aligned} \tag{1}$$

where $m_{n0} \equiv m_{n\mathrm{T}}$ and $m_{n1} \equiv m_{nX}$ denote the coefficients which we call masses of the particles. The coefficient $m_{n\mathrm{T}}$ is the temporal mass of the n^{th} particle and m_{nX} is the appropriate spatial mass. The coefficient $M_\mu = m_{1\mu} + m_{2\mu}$ can be called the total mass of this two-particle system.

Using these variables, one can introduce the total linear momentum $\hat{P}_\mu$

$$\hat{P}_\mu \equiv i\hbar \frac{\partial}{\partial \xi^\mu} = \hat{p}_{1\mu} + \hat{p}_{2\mu} \tag{2}$$

and the operator of the relative momentum $\hat{q}_\mu$

$$\hat{q}_\mu = i\hbar\frac{\partial}{\partial x^\mu} = \frac{1}{M_\mu}\left[-m_{2\mu}\hat{p}_{1\mu} + m_{1\mu}\hat{p}_{2\mu}\right] . \tag{3}$$

2.1. Simplified generator of the projection evolution for a two-particle system

For two-particle spinless system, $A = 2$, the simplified quantum evolution generator[1] expanded in linear momenta up to the second order, can be written as

$$\begin{aligned}\hat{\mathbb{W}}(\tau) \;=\; & \sum_{n=1}^{A}\sum_{\mu} a_{n\mu}(\tau)\hat{p}_{n\mu} + \sum_{n_1,n_2=1}^{A}\sum_{\mu,\nu} b_{n_1n_2;\mu\nu}(\tau)\hat{p}_{n_1\mu}\hat{p}_{n_2\nu} \\ & - \sum_{n_1<n_2}^{A} V_{n_1n_2}\left(x_{n_1}, x_{n_2}\right) + \text{h.c.}\end{aligned} \tag{4}$$

The requirement of the spatial rotational and translational symmetries simplifies this expression. After choosing the coordinates of the principal axis of the quadratic form in linear momenta, one gets

$$\begin{aligned}\hat{\mathbb{W}}(\tau) \;=\; & \sum_{n=1}^{A} a_{n0}(\tau)\hat{p}_{n0} + \sum_{n}^{A}\sum_{\mu} b_{n;\mu}(\tau)\hat{p}_{n\mu}\hat{p}_{n\mu} \\ & - \sum_{n_1<n_2}^{A} V_{n_1n_2}(x_{n_1}, x_{n_2}) + \text{h.c.}\end{aligned} \tag{5}$$

In this way, we also require that the interaction term $\sum_{n_1<n_2}^{A} V_{n_1n_2}(x_{n_1}, x_{n_2})$ has to be invariant with respect to spatial rotations.

One needs to note that the evolution generators are defined up to transformations which leave the eigenspaces of these generators invariant. Particularly $\hat{\mathbb{W}}$ and $c\hat{\mathbb{W}}$, where c is an arbitrary non-zero real number, determine the same projection evolution operators. This property allows to choose one of the coefficients arbitrarily. Let us assume, in analogy to the Schrödinger equation, that $a_{n0}(\tau) = a_{\mathrm{T}}(\tau) = 1$. Similarly, one can expect that the coefficients $b_{n;\mu}(\tau)$ are inversely proportional to the masses with the appropriate signs

$$\hat{\mathbb{W}}(\tau) = \sum_{n=1}^{A}\left\{\hat{p}_{n0} + \frac{\hat{p}_{n0}^2}{2m_{n0}} - \sum_{\mu=1}^{\dim(\boldsymbol{X})}\frac{\hat{p}_{n\mu}^2}{2m_{n\mu}}\right\} - \sum_{n_1<n_2}^{A} V_{n_1n_2}(x_{n_1}, x_{n_2}) . \tag{6}$$

[1] The projection operators projecting on eigenspaces of the evolution generator are the evolution operators.

In what follows, we work in a two-dimensional space-time, *i.e.*, we consider only one temporal and one spatial coordinate. The generalization to more spatial dimensions is straightforward. In our case, the transformation to the relative coordinates (1) gives

$$\hat{\mathbb{W}}(\tau) = \hat{P}_0 + \frac{\left(\hat{P}_0\right)^2}{2M_{\mathrm{T}}} - \frac{\left(\hat{P}_1\right)^2}{2M_X} + \frac{(\hat{q}_0)^2}{2m_{\mathrm{T}}} - \left[\frac{(\hat{q}_1)^2}{2m_X} + V\left(\tau;\xi,x\right)\right], \tag{7}$$

where $\xi = (\xi^0, \xi^1)$ are the center-of-mass coordinates and $x = (x^0, x^1)$ represents relative coordinates between two particles. The coefficients M_X, M_{T} are "space" and "temporal" total masses of both particles, respectively. The coefficients m_X, m_{T} denote the reduced "spatial" and "temporal" masses.

Let us now assume that the evolution parameter is $\tau = \tau_n$, where $n = 0, 1, 2, \ldots$ The evolution parameter τ is the common parameter for every physical subsystem. It orders the evolution steps. It has no metric structure but allows for introducing a kind of quantum causality (forward and backward).

As the next step, we modify generator (7) adding two external potentials to localize particles in space and time. This is a useful step, even though these potentials break the translational symmetry.

At the beginning, we consider the evolution generator for $\tau = \tau_0$. The first modification is adding the temporal potential $U_{\mathrm{T}}(\xi^0)$. This potential represents the averaging of times of particles in the environment of our two-particle system, which we assume to be open. The second potential $U_X(\xi^1)$ is added for technical reason to avoid unnormalizable vectors in the spatial part of the center-of-mass variable ξ^1. The evolution generator takes now the form

$$\begin{aligned}\hat{\mathbb{W}}_\sigma(\tau_0) \;=\; & \left[\hat{P}_0 + \frac{\left(\hat{P}_0\right)^2}{2M_{\mathrm{T}}} + U_{\mathrm{T}}\left(\xi^0\right)\right] - \left[\frac{\left(\hat{P}_1\right)^2}{2M_X} + U_X\left(\xi^1\right)\right] \\ & + \frac{(\hat{q}_0)^2}{2m_{\mathrm{T}}} - \frac{(\hat{q}_1)^2}{2m_X} - V_\sigma\left(x^0, x^1\right),\end{aligned} \tag{8}$$

where the sign in front of the interaction term V is arbitrary. It is chosen here in such a way that, after neglecting time dependence, the traditional form $T + V$ is obtained. We add also the subscript σ which distinguishes between two situations: the particles before ($\sigma = 1$) and after ($\sigma = 2$) the decay. The interaction between particles is chosen as independent of τ.

According to the PEv formalism, any change of the environment leads to the change of τ, which generates the next step of the projection evolution. In this context, the evolution generator should be a function of τ. Even

though the time evolution is a stochastic process, the total temporal linear momentum P_0, which is responsible for the shift of the evolution generator (8), seems to be a non-negative number for both particles. Thus, $\hat{P}_0$ determines the arrow of time. This suggests the existence of an averaged characteristic time interval t_{E} generated by the environment, which gives an averaged shift $\tau_0 \to \tau_n$ of the evolution generator

$$\begin{aligned}
\hat{\mathbb{W}}_\sigma(\tau_n) &= e^{int_{\mathrm{E}}\hat{P}_0/\hbar}\hat{\mathbb{W}}_\sigma(\tau_0)e^{-int_{\mathrm{E}}\hat{P}_0/\hbar} \\
&= \left[\hat{P}_0 + \frac{\left(\hat{P}_0\right)^2}{2M_{\mathrm{T}}} + e^{int_{\mathrm{E}}\hat{P}_0/\hbar}U_{\mathrm{T}}\left(\xi^0\right)e^{-int_{\mathrm{E}}\hat{P}_0/\hbar}\right] \\
&\quad -\left[\frac{\left(\hat{P}_1\right)^2}{2M_X} + U_X\left(\xi^1\right)\right] + \frac{\left(\hat{q}_0\right)^2}{2m_{\mathrm{T}}} - \frac{\left(\hat{q}_1\right)^2}{2m_X} - V_\sigma\left(x^0, x^1\right) \\
&= \left[\hat{P}_0 + \frac{\left(\hat{P}_0\right)^2}{2M_{\mathrm{T}}} + U_{\mathrm{T}}\left(\xi^0 - nt_{\mathrm{E}}\right)\right] - \left[\frac{\left(\hat{P}_1\right)^2}{2M_X} + U_X\left(\xi^1\right)\right] \\
&\quad + \frac{\left(\hat{q}_0\right)^2}{2m_{\mathrm{T}}} - \frac{\left(\hat{q}_1\right)^2}{2m_X} - V_\sigma\left(x^0, x^1\right), \qquad (9)
\end{aligned}$$

where $n = 0, 1, 2, \ldots$ The "external time" ξ^0 is approximately discretized by t_{E}. The time interval t_{E} is an average period of the external time during which Nature decides which of the two channels to choose: decay or not decay. Because one observes that particles are quite well-localized in time, one can assume that the temporal potential $U_{\mathrm{T}}(\xi^0)$ has a deep minimum for $\xi^0 = 0$.

Similarly, one can expect that, *e.g.*, the temporal part of the electromagnetic interaction between two particles should lead to the localization of particles in time. It means that the mesoscopic and the macroscopic worlds can be in many aspects well-described by time treated as a parameter. This leads to the confinement of both particles close to the minimum of the potential $U_{\mathrm{T}}(\xi^0)$.

3. Decay or not decay inside of a box, this is the question

In this section, we consider a very simplified model of the decay of a two-particle system. We expect two sets of preferred states coming from two generators $\hat{\mathbb{W}}_1(\tau)$ and $\hat{\mathbb{W}}_2(\tau)$. The first set is a collection of bound states of this two-particle system and the second one corresponds to the system of two spatially independent particles. To keep the simplicity of the model, the possible quantum temporal effects like time interference are to some

extend neglected by the decoupling of the spatial and temporal motions in the interaction terms of the evolution generators. This decoupling also leads to a stronger localization of the particles in time.

3.1. Spectral decomposition of the evolution generators

As it was mentioned earlier in our schematic model, we assume the interaction to be independent of τ. The only difference between both generators $\hat{\mathbb{W}}_1(\tau)$ and $\hat{\mathbb{W}}_2(\tau)$ is in the interaction terms $V_\sigma(x^0, x^1)$, for $\sigma = 1, 2$. To find the set of the evolution operators, one needs to get the spectral decompositions of both generators

$$\hat{\mathbb{W}}_\sigma(\tau_n) = \sum_\alpha w_\alpha^{(\sigma)} \mathbb{E}(\tau_n; \sigma, \alpha)\,, \tag{10}$$

where the projection operators $\mathbb{E}(\tau_n; \sigma, \alpha)$ play the role of the evolution operators. The spectral decomposition of (10) can be found by solving the eigenproblems of the generators $\hat{\mathbb{W}}_\sigma(\tau_n)$

$$\hat{\mathbb{W}}_\sigma(\tau_n)\psi_{\alpha\beta}^{(\sigma)}\left(\tau_n; \xi^0, \xi^1, x^0, x^1\right) = w_\alpha^{(\sigma)}\psi_{\alpha\beta}^{(\sigma)}\left(\tau_n; \xi^0, \xi^1, x^0, x^1\right)\,, \tag{11}$$

where $\sigma = 1, 2$ and the eigenvalues $w_\alpha^{(\sigma)}$ are, in general, degenerated.

The projection operators $\mathbb{E}(\tau_n; \sigma, \alpha)$ can be explicitly written as

$$\mathbb{E}(\tau_n; \sigma, \alpha) = \sum_\beta \left|\psi_{\alpha\beta}^{(\sigma)}(\tau_n)\right\rangle \left\langle\psi_{\alpha\beta}^{(\sigma)}(\tau_n)\right|\,. \tag{12}$$

The external clock is here simulated by the shifts of the potential $U_{\mathrm{T}}(\xi^0)$, which localizes the particles at the vicinity of the temporal center of mass ξ^0. This temporal potential is assumed here to be the harmonic oscillator potential

$$U_{\mathrm{T}}\left(\xi^0\right) = \tfrac{1}{2} M_{\mathrm{T}} \omega_{\mathrm{T}}^2 \left(\xi^0\right)^2\,. \tag{13}$$

Every tick of the external clock is treated as a new step of the projection evolution. In other words, we consider the quantum projection evolution in which the two-particle system moving to the next step of its evolution either remains bounded or decays.

The technical potential $U_X(\xi^1)$ is added to avoid problems with the continuous spectrum. In the following, we use

$$U_X\left(\xi^1\right) = \tfrac{1}{2} M_X \omega_X^2 \left(\xi^1\right)^2 \tag{14}$$

with a very small omega, $\omega_X \sim 0$. The interaction we consider is the sum of the temporal and spatial interaction terms

$$V_\sigma\left(x^0, x^1\right) = -V_\sigma^{\mathrm{T}}\left(x^0\right) + V_\sigma^X\left(x^1\right)\,, \tag{15}$$

where $V_\sigma^{\mathrm{T}}(x^0)=V_\infty(a_{\mathrm{T}};x^0)$, $V_1^X(x^0)=V_\infty(a_X;x^1)$, and $V_2^X(x^0)=V_\infty(L;x^1)$. Here, $V_\infty(a;x^\mu)$ denotes the infinite well type interaction defined as

$$V_\infty(a;x^\mu)=\begin{cases}0\,, & \text{for } |x^\mu|<a\,,\\ \infty\,, & \text{for } |x^\mu|\geq a\,.\end{cases} \tag{16}$$

The parameters $a=a_{\mathrm{T}},a_X$ simulate the range of the temporal and spatial interactions, respectively. The parameter $L>a_X$ describes the size of the free spatial box in which our two-particle system decays. This box is simulated by the upper limit L of the distance between the free particles after the decay.

The eigenvalues of generators (11) can thus be written as

$$w_\alpha^{(\sigma)}=\lambda_{\kappa_0}^{(\mathrm{T})}-\lambda_{\kappa_1}^{(X)}+\lambda_{\pi_{\mathrm{T}},s_{\mathrm{T}}}^{(\mathrm{T},\sigma)}-\lambda_{\pi_X,s_X}^{(X,\sigma)}\,, \tag{17}$$

where different values of α describe distinct eigenvalues. The corresponding eigenvectors are

$$\psi_{\alpha\beta}^{(\sigma)}\left(\tau_n;\xi^0,\xi^1,x^0,x^1\right)=\chi_{\kappa_0}^{(\mathrm{T})}\left(\xi^0-nt_{\mathrm{E}}\right)\,\chi_{\kappa_1}^{(X)}\left(\xi^1\right)\,\phi_{\pi_{\mathrm{T}},s_{\mathrm{T}}}^{(\mathrm{T},\sigma)}\left(x^0\right)\,\phi_{\pi_X,s_X}^{(X,\sigma)}\left(x^1\right)\,. \tag{18}$$

Here, $n=0,1,2,3,\ldots$ and $\kappa_0,\kappa_1,\pi_{\mathrm{T}},s_{\mathrm{T}},\pi_X,s_X$ are the harmonic oscillator quantum numbers of the global temporal motion and the global spatial motion, parity of the temporal relative motion function and its quantum number, the parity of the spatial relative motion function and its quantum number, respectively. The quantum number $\beta=\{\kappa_0,\kappa_1,\pi_{\mathrm{T}},s_{\mathrm{T}},\pi_X,s_X\}$ distinguishes the eigenvectors belonging to a given eigenvalue $w_\alpha^{(\sigma)}$. The global temporal motion is described by the eigenfunctions

$$\begin{aligned}\chi_{\tau_n,\kappa_0}^{(\mathrm{T})}\left(\xi^0\right) &\equiv \chi_{\kappa_0}^{(\mathrm{T})}\left(\xi^0-nt_{\mathrm{E}}\right)\\ &= e^{int_{\mathrm{E}}\hat{P}_0}\chi_{\kappa_0}^{(\mathrm{T})}\left(\xi^0\right)=e^{iM_{\mathrm{T}}\left(\xi^0-nt_{\mathrm{E}}\right)/\hbar}u_{\kappa_0}\left(\eta_{\mathrm{T}};\xi^0-nt_{\mathrm{E}}\right)\,,\end{aligned} \tag{19}$$

with $n=0,1,2,3,\ldots$ and the corresponding eigenvalues

$$\lambda_{\kappa_0}^{(\mathrm{T})}=\hbar\omega_{\mathrm{T}}\left(\kappa_0+\frac{1}{2}\right)-\frac{M_{\mathrm{T}}}{2}\,. \tag{20}$$

The functions $u_{\kappa_0}(\eta_{\mathrm{T}};\xi^0)$ are the eigenfunctions of the one-dimensional harmonic oscillator,

$$u_l(\eta;x)=\sqrt{\frac{\eta}{\sqrt{\pi}2^l l!}}H_l(\eta x)\exp\left(-\frac{\eta^2x^2}{2}\right)\,, \tag{21}$$

where $\eta = \sqrt{m\omega/\hbar}$. The global spatial motion is determined here by the harmonic oscillator functions

$$\chi^{(X)}_{\kappa_1}\left(\xi^1\right) = u_{\kappa_1}\left(\eta_X;\xi^1\right)\,, \tag{22}$$

and the eigenvalues

$$\lambda^{(X)}_{\kappa_1} = \hbar\omega_X\left(\kappa_1+\tfrac{1}{2}\right)\,. \tag{23}$$

The eigenfunctions of the relative motion are given by the eigenvalue problem with the infinite well potential. We denote these eigenfunctions by

$$v_s(\varpi,a;x^\mu)=\begin{cases}\frac{1}{\sqrt{2a}}\cos\left(\frac{\pi(2s+1)}{2a}x^\mu\right)\delta(|x^\mu|<a)\,, & \text{for } \varpi=+1\,,\ s=0,1,2,\dots\,,\\ \frac{1}{\sqrt{2a}}\sin\left(\frac{\pi s}{a}x^\mu\right)\delta(|x^\mu|<a)\,, & \text{for } \varpi=-1\,,\ s=1,2,\dots\,,\end{cases} \tag{24}$$

where $\delta(|x|<a)=1$ for $|x|<a$ and zero otherwise, and $\varpi=\pm1$ denotes the parity of the eigenfunctions. The corresponding eigenvalues are

$$\lambda^{\infty}_{a;\varpi,s} = \frac{\hbar^2\pi^2}{2ma^2}\begin{cases}\left(\frac{2s+1}{2}\right)^2 & \text{for } \varpi=+1\,,\\ s^2 & \text{for } \varpi=-1\,.\end{cases} \tag{25}$$

Using these functions, the temporal relative motion is described by

$$\begin{aligned}\phi^{(\mathrm{T},\sigma)}_{\pi_\mathrm{T},s_\mathrm{T}}\left(x^0\right) &= v_{s_\mathrm{T}}\left(\pi_\mathrm{T},a_\mathrm{T};x^0\right)\,,\\ \lambda^{(\mathrm{T},\sigma)}_{\pi_\mathrm{T},s_\mathrm{T}} &= \lambda^{\infty}_{a_\mathrm{T};\pi_\mathrm{T},s_\mathrm{T}}\,,\end{aligned} \tag{26}$$

and the spatial relative motion before ($\sigma=1$) and after ($\sigma=2$) the decay is given by

$$\phi^{(X,\sigma)}_{\pi_X,s_X}\left(x^1\right) = \begin{cases}v_{s_X}\left(\pi_X,a_X;x^1\right) & \text{for } \sigma=1\,,\\ v_{s_X}\left(\pi_X,L;x^1\right) & \text{for } \sigma=2\,,\end{cases} \tag{27}$$

$$\lambda^{(X,\sigma)}_{\pi_X,s_X} = \begin{cases}\lambda^{\infty}_{a_X;\pi_X,s_X} & \text{for } \sigma=1\,,\\ \lambda^{\infty}_{L;\pi_X,s_X} & \text{for } \sigma=2\,.\end{cases} \tag{28}$$

3.2. Evolution operators and the decay probability

The evolution operators constructed as projections onto eigenspaces of the generators $\hat{\mathsf{W}}_\sigma(\tau_n)$ have the standard form (12)

$$\begin{aligned}\mathbb{E}(\tau_n;\sigma,\alpha) = &\sum_{\kappa_0,\kappa_1,s_\mathrm{T},s_X}\delta\left(w^{(\sigma)}_\alpha=\lambda^{(\mathrm{T})}_{\kappa_0}-\lambda^{(X)}_{\kappa_1}+\lambda^{(\mathrm{T},\sigma)}_{\pi_\mathrm{T},s_\mathrm{T}}-\lambda^{(X,\sigma)}_{\pi_X,s_X}\right)\\ &\times\left|\psi^{(\sigma)}_{\alpha\kappa_0,\kappa_1,\pi_\mathrm{T},s_\mathrm{T},\pi_X,s_X}(\tau_n)\right\rangle\left\langle\psi^{(\sigma)}_{\alpha\kappa_0,\kappa_1,\pi_\mathrm{T},s_\mathrm{T},\pi_X,s_X}(\tau_n)\right|\,,\end{aligned} \tag{29}$$

where $\delta(a=b)=1$ if $a=b$ and zero otherwise.

In this model, we have a set of events represented by two orthogonal decompositions of unity $\mathbb{E}(\tau;\sigma,\alpha)$, $\sigma = 1,2$. The renormalized operators belonging to both sets can be considered as the evolution operators

$$\mathbb{E}'(\tau_n;\sigma,\alpha) = \sqrt{p_\sigma}\,\mathbb{E}(\tau_n;\sigma,\alpha)\,, \tag{30}$$

where $p_1, p_2 \geq 0$ and $p_1 + p_2 = 1$ are the maximal probabilities of choosing either the first channel (not decay) or the second channel (decay). The operators

$$\mathbb{E}'(\tau_n;\sigma,\alpha)^\dagger \mathbb{E}'(\tau_n;\sigma,\alpha) = p_\sigma \mathbb{E}(\tau_n;\sigma,\alpha) \tag{31}$$

give the positive operator valued measure (POVM) which defines the required quantum probability measure determining the probability of choosing the next state for the following step of the evolution.

Assuming that the density operator $\rho(\tau_n;\tilde{\sigma},\tilde{\alpha})$ represents the state of the system for $\tau = \tau_n$, the next state for $\tau = \tau_{n+1}$ is chosen randomly according to PEv as

$$\rho(\tau_{n+1};\sigma,\alpha) = \frac{\mathbb{E}'(\tau_{n+1};\sigma,\alpha)\rho(\tau_n;\tilde{\sigma},\tilde{\alpha})\mathbb{E}'(\tau_{n+1};\sigma,\alpha)}{\mathrm{Tr}\left(\mathbb{E}'(\tau_{n+1};\sigma,\alpha)\rho(\tau_n;\tilde{\sigma},\tilde{\alpha})\mathbb{E}'(\tau_{n+1};\sigma,\alpha)\right)}\,, \tag{32}$$

with the probability distribution

$$\mathrm{pev}(\tau_{n+1};(\tilde{\sigma},\tilde{\alpha}) \to (\sigma,\alpha)) = p_\sigma \mathrm{Tr}\left(\mathbb{E}(\tau_{n+1};\sigma,\alpha)\rho(\tau_n;\tilde{\sigma},\tilde{\alpha})\right)\,. \tag{33}$$

The total temporal $\hat{P}_0$ and spatial $\hat{P}_1$ momenta are conserved quantities. In our model, this property is broken by the generators $\hat{\mathbb{W}}_\sigma(\tau_n)$ because of the external potentials U_T and U_X. To fulfill the conservation law on average, however, we keep the quantum numbers κ_0 and κ_1 fixed during the evolution process. One can find that for the special case of a decay from the state with positive parity and $\kappa_0 = 0$, for either a very small elementary clock tick t_E or a wide temporal potential U_T, *i.e.*, for $\eta_\mathrm{T} t_\mathrm{E} \sim 0$, the probability of changing κ_0 is very small. The same effect can be observed for $\eta_\mathrm{T} t_\mathrm{E} \gtrsim 5$. In our example of the application of this model, we will assume $\tilde{\kappa}_0 = \kappa_0$.

To neglect the temporal interaction effects, which require more detailed analysis and are out of scope of this article, we fix the range a_T of the temporal interaction. Assuming, in addition, that the model parameters are chosen so that the generators have no degeneracy in their spectra, the evolution operators reduce to simple, one-dimensional operators

$$\begin{aligned}&\mathbb{E}'(\tau_{n+1};\sigma,\kappa_0,\kappa_1,\pi_\mathrm{T},s_\mathrm{T},\pi_X,s_X) = \\ &\left|\chi^{(\mathrm{T})}_{\tau_{n+1},\kappa_0}\chi^{(X)}_{\kappa_1}\phi^{(\mathrm{T},\sigma)}_{\pi_\mathrm{T},s_\mathrm{T}}\phi^{(X,\sigma)}_{\pi_X,s_X}\right\rangle \sqrt{p}_\sigma \left\langle \chi^{(\mathrm{T})}_{\tau_{n+1},\kappa_0}\chi^{(X)}_{\kappa_1}\phi^{(\mathrm{T},\sigma)}_{\pi_\mathrm{T},s_\mathrm{T}}\phi^{(X,\sigma)}_{\pi_X,s_X}\right|\,.\end{aligned} \tag{34}$$

Assume now that for $\tau = \tau_n$, the particles are in an eigenstate of the generator $\hat{\mathbb{W}}_{\tilde{\sigma}}(\tau_n)$: $|\chi^{(\mathrm{T})}_{\tau_n,\tilde{\kappa}_0}\chi^{(X)}_{\tilde{\kappa}_1}\phi^{(\mathrm{T},\tilde{\sigma})}_{\tilde{\pi}_\mathrm{T},\tilde{s}_\mathrm{T}}\phi^{(X,\tilde{\sigma})}_{\tilde{\pi}_X,\tilde{s}_X}\rangle$. The transition probability (33) to a new state is now given by the simple scalar products

$$\begin{aligned}&\mathrm{pev}(\tau_{n+1};(\tilde{\sigma},\tilde{\kappa}_0,\tilde{\kappa}_1,\tilde{\pi}_\mathrm{T},\tilde{s}_\mathrm{T},\tilde{\pi}_X,\tilde{s}_X)\to(\sigma,\kappa_0,\kappa_1,\pi_\mathrm{T},s_\mathrm{T},\pi_X,s_X))\\&=\delta_{\kappa_1\tilde{\kappa}_1}\delta_{\pi_\mathrm{T}\tilde{\pi}_\mathrm{T}}\delta_{\pi_X\tilde{\pi}_X}\delta_{s_\mathrm{T}\tilde{s}_\mathrm{T}}p_\sigma\left|\left\langle\chi^{(\mathrm{T})}_{\tau_1,\kappa_0}\middle|\chi^{(\mathrm{T})}_{\tau_0,\tilde{\kappa}_0}\right\rangle\right|^2\left|\left\langle\phi^{(X,\sigma)}_{\pi_X,s_X}\middle|\phi^{(X,\tilde{\sigma})}_{\pi_X,\tilde{s}_X}\right\rangle\right|^2.\end{aligned}\tag{35}$$

The last equation is obtained using the unitarity of transformation (19). Note that probability (35) is, in this schematic model, independent of the evolution parameter τ.

The bound system can decay to different free particle states with different probabilities. In Fig. 1, examples of the transition probabilities $((\tilde{\sigma}=1,\tilde{\kappa}_0=0,\tilde{\kappa}_1,\tilde{\pi}_\mathrm{T},\tilde{s}_\mathrm{T},\tilde{\pi}_X=+1,\tilde{s}_X=0)\to(\sigma=2,\kappa_0=0,\kappa_1,\pi_\mathrm{T},s_\mathrm{T},\pi_X=+1,s_X=1,5,9,13)))$ are presented as functions of $\gamma=\frac{a_X}{L}$, which is the ratio of the interaction range a_X in the position space and the spatial size of the box L. One clearly sees that the terms with lower s_X dominate, with the leading role played by the $s_X=1$ contribution. This term drops to zero as $\gamma\to 1$.

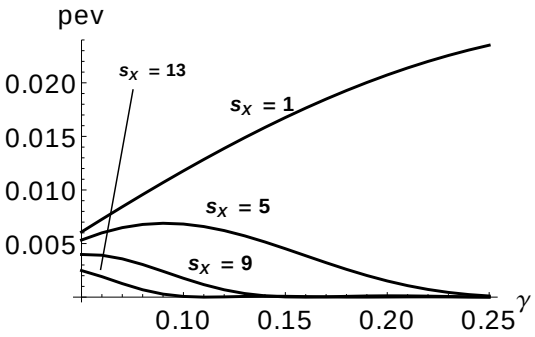

Fig. 1. A few examples of the transition probabilities from the bound to unbound states, as functions of $\gamma = a_X/L$.

The most interesting is the total decay probability in one evolution step from a given fixed state to all allowed states. To calculate the total decay probability, one needs to sum up the partial probabilities (35) over the allowed final states, *i.e.*, the states with lower than initial energy represented by the spatial Hamiltonian.

Assume that at the evolution step τ_n, we still have the two-particle system bounded. Because, in this case, the evolution operator projects on the eigenspaces of the generator $\hat{\mathbb{W}}_1(\tau_n)$, the two-particle system has to be in an eigenstate of $\hat{\mathbb{W}}_1(\tau_n)$

$$\Psi_{n,\tilde{\nu}}\left(\tau_n;\xi^0,\xi^1,x^0,x^1\right)=\psi^{(\tilde{\sigma}=1)}_{\tilde{\kappa}_0,\tilde{\kappa}_1,\pi_\mathrm{T},\tilde{s}_\mathrm{T},\pi_X,\tilde{s}_X}\left(\tau_n;\xi^0,\xi^1,x^0,x^1\right),\tag{36}$$

where $\tilde{\nu} = (\tilde{\sigma}, \tilde{\kappa}_0, \tilde{\kappa}_1, \pi_{\mathrm{T}}, \tilde{s}_{\mathrm{T}}, \pi_X, \tilde{s}_X)$. The Hamiltonians which determine the set of allowed final states, *i.e.*, states with the appropriate energies, are of the form of

$$H(\sigma) = \frac{(\hat{q}_1)^2}{2m_X} + V_\sigma\left(x^1\right) . \tag{37}$$

In our case, the eigenstates of the evolution generators are also eigenstates of the Hamiltonians $H(\sigma)$ and it is enough to compare the appropriate eigenvalues to find the set of allowed final quantum numbers. Because we assumed no changes in the temporal part of the interaction between particles, for every initial quantum number $\tilde{s}_X$, we need to find the set of the final quantum numbers s_X. Using formulae (25), one gets the maximal allowed value of s_X as a function of initial $\tilde{s}_X$

$$s_{X,\max} = \begin{cases} \frac{1}{\gamma}\tilde{s}_X & \text{for } \pi_X = -1\,, \\ \frac{1}{2}\left(\frac{1}{\gamma}(2\tilde{s}_X + 1) - 1\right) & \text{for } \pi_X = +1\,. \end{cases} \tag{38}$$

The total decay probability from the state $\Psi_{n,\tilde{\nu}}$, given by Eq. (36), to the set of allowed states $\Psi_{n+1,\nu}$ is

$$\begin{aligned} \mathrm{pev}_{\mathrm{D}} \;=\; & \delta_{\kappa_1\tilde{\kappa}_1}\delta_{\pi_{\mathrm{T}}\tilde{\pi}_{\mathrm{T}}}\delta_{\pi_X\tilde{\pi}_X}\delta_{s_{\mathrm{T}}\tilde{s}_{\mathrm{T}}} \\ & \times p_2 \left|\left\langle \chi^{(\mathrm{T})}_{\tau_1,\kappa_0}\middle|\chi^{(\mathrm{T})}_{\tau_0,\tilde{\kappa}_0}\right\rangle\right|^2 \sum_{s_X \le s_{X,\max}} \left|\left\langle \phi^{(X,\sigma=2)}_{\pi_X,s_X}\middle|\phi^{(X,\tilde{\sigma}=1)}_{\pi_X,\tilde{s}_X}\right\rangle\right|^2 . \end{aligned} \tag{39}$$

As it was mentioned earlier, this elementary decay probability does not depend on the evolution parameter τ_n explicitly. Using the above observations we see that, in our case, every step of the projection evolution leads to a constant probability $b_{\mathrm{D}} = \mathrm{pev}_{\mathrm{D}}(\mathrm{g.s.} \to (\sigma = 2, \kappa_0 = 0, \kappa_1, \pi_{\mathrm{T}} = +1, s_{\mathrm{T}}, \pi_X = +1, s_X = 0))$ for the decay and $(1 - b_{\mathrm{D}})$ for remaining in the undecayed state. In Fig. 2 the total probability decay from a given evolution step to the next one is plotted as a function of the γ parameter. The decay goes from the positive parity state with $\tilde{s}_X = 0$, and $\tilde{\kappa}_0 = 0$.

The minimal value of the parameter $\gamma = 0$ corresponds either to the zero-range interaction in the finite-size box ($L < \infty$) or to the infinite box ($L = \infty$). The maximal value of $\gamma = 1$ describes the maximal-range interaction in the box — it simulates a kind of a long range interaction. In Fig. 2, the total decay shows the influence of the size of the box on the decay probability — more free space for the decaying particles implies larger decay probability. The jumps of the decay probability curve correspond to the change of numbers of allowed states to which the system can decay. Both features seem to be a characteristic behaviour of the decay in a spatial box.

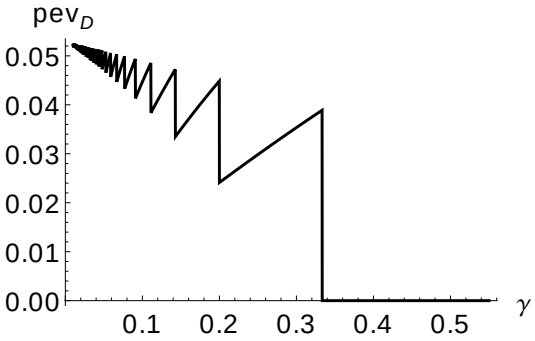

Fig. 2. The total decay probability as a function of $\gamma = a_X/L$, from the bound to all allowed unbound states.

4. Summary

In this paper, we considered a very schematic model of a decay of a two-particle system. We have used an extension of the standard quantum mechanics called the projection evolution (PEv), which allows to treat time on the same footing as the spatial coordinates. In our model, however, we have strongly reduced the most possible temporal effects, which deserve a more advanced analysis. On the other hand, the external time (external clock) determined by the environment of our two-particle system is considered. We have studied the competition between two processes: the evolution of our system in the external time without and with the decay. In our calculations, we have considered the effects of the size of the box *versus* the interaction range, which turned out to be an important factor for the decay.

REFERENCES

[1] U. Hauser, W. Neuwirth, N. Thesen, *Phys. Lett. A* **49**, 57 (1974).

[2] F. Lindner *et al.*, *Phys. Rev. Lett.* **95**, 040401 (2005).

[3] J.G. Muga, R. Sala Mayato, I.L Egusquiza (Eds.), *Time in Quantum Mechanics*, Lecture Notes in Physics, vol. 734, Springer-Verlag, 2008, doi:10.1007/978-3-540-73473-4.

[4] J.G. Muga, A. Ruschhaupt, A. del Campo (Eds.), *Time in Quantum Mechanics — Vol. 2*, Lecture Notes in Physics, vol. 789, Springer-Verlag, 2009, doi:10.1007/978-3-642-03174-8.

[5] L.P. Horwitz, *Relativistic Quantum Mechanics*, Fundamental Theories of Physics, vol. 180, Springer Science+Vussines Media, 2015.

[6] A. Góźdź, K. Rybak, A. Pędrak, M. Góźdź, *Acta Phys. Pol. B Proc. Suppl.* **8**, 591 (2015).

[7] A. Góźdź, K. Stefańska, *J. Phys.: Conf. Ser.* **104**, 012007 (2008).
[8] A. Góźdź, M. Dębicki, K. Stefańska, *Phys. At. Nucl.* **71**, 892 (2008).
[9] A. Góźdź, M. Dębicki, *Phys. At. Nucl.* **70**, 529 (2007).
[10] A. Góźdź, K. Stefańska, *Int. J. Mod. Phys. E* **17**, 217 (2008).
[11] M. Dębicki, A. Góźdź, K. Stefańska, *Int. J. Mod. Phys. E* **16**, 616 (2007).
[12] M. Góźdź, A. Góźdź, *Phys. Scr.* **89**, 054010 (2014).