COMPARISON BETWEEN DIRAC AND REDUCED QUANTIZATION IN LQG-MODELS WITH KLEIN–GORDON SCALAR FIELDS*

K. Giesel†, A. Oelmann‡

Institute for Quantum Gravity
Friedrich-Alexander University Erlangen–Nürnberg
Staudtstr. 7, 91058 Erlangen, Germany

(Received April 18, 2017)

In this letter, we discuss a comparison between two scalar field models that have been recently introduced in the context of loop quantum gravity. The scalar fields play the role of so-called reference fields that allow to construct Dirac observables for general relativity and introduce a notion of physical spatial and time coordinates respectively. One of the models uses Dirac quantization, the other one reduced phase space quantization. We want to compare the physical sector of both quantum theories and discuss their similarities and differences with a particular focus on their quantum dynamics.

DOI:10.5506/APhysPolBSupp.10.339

1. Introduction

The canonical approach of loop quantum gravity (LQG) aims to formulate a canonical quantization of general relativity (GR). For any canonical quantization of a given classical field theory, we have to make some choices in order to obtain the final quantum theory. The first necessary choice is the classical theory to start with and this is GR in its Hamiltonian form. Instead of the ADM formulation, we consider GR described in terms of the so-called Ashtekar variables which are a Lie(SU(2))-algebra valued one form as the configuration variable A and a Lie(SU(2))-algebra valued vector density E for the conjugate momenta. This brings an additional SU(2)-gauge freedom into GR that can be understood as an extension of the usual ADM phase

* Talk presented by K. Giesel at the 3rd Conference of the Polish Society on Relativity, Kraków, Poland, September 25–29, 2016.

† kristina.giesel@gravity.fau.de

‡ almut.oelmann@gravity.fau.de

space, which can be recovered if the Gauss constraint is satisfied. Thus, the set of constraints the theory has are the spatial diffeomorphism constraint, the Hamiltonian constraint, known from the ADM framework, as well as the Gauss constraint. In order to go over to the quantum theory, one considers the so-called holonomy-flux algebra that can be obtained by a specific smearing of the connection and momentum variables. This is the first main assumption of LQG because it fixes the classical variables the quantum theory will be based on. The second main assumption determining the final form of the quantum theory is the choice of a representation that allows to represent the classical phase space variables as linear operators on some Hilbert space. Here, we choose the Ashtekar–Lewandowski (AS) representation that is widely used in the context of LQG, see, for instance, [6] for more details. Next to the two main assumptions for a quantum field theory discussed above for GR as our classical starting point, we have a third main assumption that also determines the properties of the quantum theory. This is the choice how we treat the constraints that occur in the Hamiltonian formulation of GR. To obtain the corresponding quantum theory that encodes this set of first class constraints, we have two options: Either we can use Dirac quantization or we can use reduced phase space quantization. Both methods will be briefly summarized in the next section. Both routes have been followed and lead to several recent so-called deparametrized models for loop quantum gravity, see, for instance, [1–5]. In this letter, we want to compare the models obtained in [3] and [5].

2. Dirac quantization and reduced phase space quantization

The Dirac quantization and reduced phase space quantization approaches differ in the way how one handles the constraints when going over from the classical theory to the corresponding quantum theory. In the case of Dirac quantization, one quantizes the entire kinematical phase space. What one obtains after quantization is the so-called kinematical Hilbert space $\mathcal{H}_{\text{kin}}$ that involves besides the physical degrees of freedom (dof) also still gauge dof. At the classical level, the physical dof can be obtained by reducing with respect to the (classical) constraints which we denote by the set $\{c_I\}$. In the Dirac approach, this is carried over to the quantum theory where one implements the classical constraints as operators on $\mathcal{H}_{\text{kin}}$. One requires that physical states are annihilated by all constraint operators, that is $\hat{c}_I\psi_{\text{phys}} = 0$ for all I. Once the set of physical states has been determined, one has to define an inner product on this set leading to the physical Hilbert space $\mathcal{H}_{\text{phys}}$ that one is finally interested in. On the other hand, if one follows reduced phase space quantization, the constraints are already reduced at the classical level. By this, the kinematical algebra is replaced by the al-

gebra of observables that encodes the physical dof only. Then, one looks for representations of the observable algebra providing a direct access to the physical Hilbert space $\mathcal{H}_{\text{phys}}$. Since the constraints have been reduced at the classical level, there is no dynamics described by quantum constraint equations in this approach. However, the dynamics of the observables on the reduced phase space is generated by a so-called physical Hamiltonian, that, in contrast to the constraints, does not vanish on the physical sector of the theory. Both procedures have their advantages and disadvantages. In the Dirac approach, solving the constraints corresponds to finding solutions of possibly complicated operator equations, whereas in the reduced quantization, in general, the algebra of observables can have a much more complicated structure than the associated kinematical one. Hence, it can be hard to find representations thereof. Moreover, for a given theory, one does not have to strictly follow one or the other approach, but can also combine the two by solving a part of the constraints at the classical level and the remaining ones at the quantum level. For LQG, both routes were taken and the reduced quantization requires to construct the algebra of observables which will be briefly discussed in the next subsection.

2.1. Dirac observables for general relativity

As mentioned above, for GR formulated in terms of Ashtekar variables (A, E), we have to consider the Hamiltonian, spatial diffeomorphism and Gauss constraints denoted by c, c_a and g_j, respectively, where $a = 1, 2, 3$ denotes spatial indices and $j = 1, 2, 3$ are Lie algebra indices. In all models discussed in this letter, the Gauss constraint is solved at the quantum level because it can be easily solved using standard techniques from the lattice gauge theory. Hence, following the reduced quantization approach, we need to construct observables with respect to the Hamiltonian and spatial diffeomorphism constraint. These Dirac observables O are quantities on phase space that satisfy

$$\{O, c_a\} = 0 \qquad \text{and} \qquad \{O, c\} = 0\,.$$

The construction of these observables can be done in the framework of the so-called relational formalism [7–9] where we introduce reference fields that are used to define physical time and spatial coordinates and with respect to which the dynamics of the remaining dof is described. We choose one reference field for each constraint denoted by $T := \{T^I\}$ with $I = 0, 1, 2, 3$. As shown in [8], there exists a map that sends each phase space function f to its associated observable denoted by $O_{f,T}(\sigma^j, \tau)$. The argument τ is the value that the time reference field T^0 takes, whereas the values σ^j with $j = 1, 2, 3$ are those values that the spatial reference fields T^j take. A brief summary

with more details about the observable map can be found in [5, 10]. The construction requires that the reference fields and the constraints satisfy, at least weakly, $\{T^I, c_J\} \simeq \delta^I_J$. This is usually obtained by going over to an equivalent set of constraints for which this property is satisfied. The dynamics of the observables cannot certainly be generated by the canonical Hamiltonian because we have by construction $\{O_{f,T}, H_{\text{can}}\} \simeq 0$ and thus only a trivial dynamics would be allowed. Instead, the dynamics is generated by a so-called physical Hamiltonian H_{phys} that does not vanish on the physical phase space. The corresponding Hamiltonian equations are given by

$$\frac{O_{A,T}(\tau)}{\mathrm{d}\tau} = \{O_{A,T}(\tau), H_{\text{phys}}\}\,, \qquad \frac{O_{E,T}(\tau)}{\mathrm{d}\tau} = \{O_{E,T}(\tau), H_{\text{phys}}\}\,.$$

In the context of the deparametrized models for LQG, either one or four reference fields are chosen. In the first case, we obtain a partially reduced phase space with respect to the Hamiltonian constraint and the spatial diffeomorphism constraints are solved via Dirac quantization. For the latter case, we obtain the fully reduced phase space. Now, in general, both kinds of models describe a quantization of GR. However, since these models differ for instance in the total number of dof, this is already a hint that a comparison of the two quantum theories might be a non-trivial step. In general, even if we would quantize exactly the same model with either Dirac or reduced quantization, it might be the case that we end up with different quantum theories. In the following, we want to compare the models obtained in [3] and [5] and discuss their similarities and differences.

3. The one Klein–Gordon scalar field model

The model introduced in [3] considers gravity and one massless Klein–Gordon (KG) scalar field described by the following action

$$S\left[g, \varphi^0\right] = \frac{1}{\kappa}\int \mathrm{d}^4X \sqrt{\det(g)}R + \frac{1}{2}\int \mathrm{d}^4X g^{\mu\nu}\varphi^0_{,\mu}\varphi^0_{,\nu}$$

with $\kappa = \frac{1}{16\pi G_N}$ and we introduced the label 0 to indicate that φ^0 is the reference field for the time coordinate. This model is a natural generalization of the APS model in loop quantum cosmology [13] to full loop quantum gravity. In order to obtain the physical sector in the quantum theory, one uses the following strategy: In [3] the constraints c_a, g_j are solved via Dirac quantization and this requires to work at the level of the Gauss and diffeomorphism invariant Hilbert space $\mathcal{H}_{\text{diff}}$. The Hamiltonian constraint has the following form

$$c = c^{\text{geo}} + \frac{1}{2}\frac{\pi_0^2}{\sqrt{q}} + \frac{1}{2}\sqrt{q}q^{ab}\varphi^0_{,a}\varphi^0_{,b} \qquad \text{with} \qquad q := \det(q)\,.$$

One solves the Hamiltonian constraint for the scalar field momentum π_0 by using the Brown–Kuchař mechanism, that is using $\varphi^0_{,a} = -\frac{c^{\rm geo}_a}{\pi_0}$. This ensures that the constraint can be written in deparametrized form[1] and obtains the equivalent constraint

$$\tilde{c} = \pi_0 - h\,, \qquad h = \sqrt{-\sqrt{q}\,c^{\rm geo} + \sqrt{q}\sqrt{(c^{\rm geo})^2 - q^{ab}c^{\rm geo}_a c^{\rm geo}_b}}\,. \tag{3.1}$$

The double square root comes from the fact that after using the Brown–Kuchař mechanism, the constraint is a fourth order polynomial in π_0. Then, one promotes the constraint to an operator on $\mathcal{H}_{\rm diff}$ and shows that due to the fact that φ^0 is chosen as a reference field for time and $\hat{\pi}_0(x) = -i\hbar\frac{\delta}{\delta\varphi^0(x)}$, the constraint equation $\hat{\tilde{c}}\psi = 0$ can be expressed as a Schrödinger equation for ψ with a physical Hamiltonian of the form of

$$\hat{H}_{\rm phys} = \int {\rm d}^3x \sqrt{-\sqrt{\hat{q}}\,\hat{c}^{\rm geo} + \sqrt{\hat{q}}\sqrt{(\hat{c}^{\rm geo})^2 - q^{ab}\widehat{c^{\rm geo}_a c^{\rm geo}_b}}}\,.$$

Since $H_{\rm phys}$ is implemented on $\mathcal{H}_{\rm diff}$, the second term in the second square root should annihilate spatially diffeomorphism invariant states[2] and hence the final Hamiltonian one works with in that model is

$$\hat{H}_{\rm phys} = \int {\rm d}^3x \sqrt{-2\sqrt{\hat{q}}\,\hat{c}^{\rm geo}}\,.$$

Being already at the quantum level following [3], one can construct quantum Dirac observables whose classical limit are the Dirac observables discussed above.

Here, we will slightly modify the model and consider the partially reduced phase space with respect to the Hamiltonian constraint already at the classical level and show that we end up with exactly the same form of the physical Hamiltonian. Instead of the Brown–Kuchař mechanism, we consider the result of [4] and allow an equivalent Hamiltonian constraint $\tilde{c}'$ that does not deparametrize, but depends on the reference field φ^0 only in a specific form — in our case via its spatial derivatives. We solve, as before, for π_0 without using the Brown–Kuchař mechanism and obtain

$$\tilde{c}' = \pi_0 - h\,, \qquad h = \sqrt{-2\sqrt{q}\,c^{\rm geo} - qq^{ab}\varphi^0_{,a}\varphi^0_{,b}}\,.$$

[1] A constraint c deparametrizes if it can be written as $c = p - h$ where h does not depend on the configuration variable associated with the momentum p.

[2] This has, strictly speaking, not be shown in [3] but just assumed to be valid for that model.

The constraint $\tilde{c}'$ satisfies $\{\varphi^0(x), \tilde{c}'(y)\} = \delta^{(3)}(x,y)$ and hence we are in the situation that we can apply the observable map and construct observables with respect to the Hamiltonian constraint $\tilde{c}'$. We do this for the elementary phase space variables (A, E) and use the property from [8] that the multi parameter family of maps $O_\tau : f \mapsto O_{f,T}(\tau)$ is a homomorphism from the commutative algebra of functions on phase space to the commutative algebra of weak Dirac observables, both with pointwise multiplication,

$$O_{f,T}(\tau) + O_{g,T}(\tau) = O_{f+g,T}(\tau)\,, \qquad O_{f,T}(\tau) O_{g,T}(\tau) = O_{fg,T}(\tau)\,.$$

Considering this, we have in our case for a generic phase space function $f(A,E)$,

$$O_{f(A,E),T}(\tau) = f(O_{A,T}(\tau), O_{E,T}(\tau))\,.$$

Using this, it can be shown that the physical Hamiltonian for the observables defined on the (partially) reduced phase space has the form of

$$H_{\rm phys} = \int \mathrm{d}^3x H(x) \qquad \text{with} \qquad H(x) = O_{h,\varphi^0} = \sqrt{-2\sqrt{Q}\, C^{\rm geo}}\,, \quad (3.2)$$

where we used the abbreviations $O_{c^{\rm geo},\varphi^0} = C^{\rm geo}$, $Q_{ab} = O_{q_{ab},\varphi^0}$ and we used that $O_{\varphi^0_{,a},\varphi^0} = 0$ [4]. The ADM 3-metric q_{ab} is understood as a function of A, E. The algebra of these elementary variables is given by [9]

$$\begin{aligned}\{O_{A(x),\varphi^0}(\tau), O_{E(y),\varphi^0}(\tau)\} &= O_{\{A(x),E(y)\}^*,\varphi^0}(\tau) = O_{\{A(x),E(y)\},\varphi^0}(\tau)\\ &= \delta^{(3)}(x,y)\,.\end{aligned}$$

Here, $\{.,.\}^*$ denotes the Dirac bracket constructed from the set of second class constraints ($g_\tau := \varphi^0 - \tau, \tilde{c}'$) which simplifies to the Poisson bracket for A, E because both of them commute with the reference field φ^0. We realize that the elementary observables satisfy a standard canonical algebra and hence a representation of this algebra can be found. Since in this model, the Gauss and diffeomorphism constraints are again solved via Dirac quantization, the physical Hilbert space $\mathcal{H}_{\rm phys}$ can be identified with the above mentioned $\mathcal{H}_{\rm diff}$. Therefore, we obtain the same physical sector in the quantum theory with the same physical Hamiltonian operator $\hat{H}_{\rm phys}$ as in [3], where there is explained in detail how $\hat{H}_{\rm phys}$ can be implemented on $\mathcal{H}_{\rm diff}$. The advantage compared to the derivation in [3] is that if we directly quantize the partially reduced phase space, we do not obtain a double square root form of the physical Hamiltonian and hence do not have to assume that certain parts of this operator vanish on spatially diffeomorphism invariant states. In the next section, we want to discuss a model with four reference fields for which only the Gauss constraint is solved via Dirac quantization.

4. The four Klein–Gordon scalar fields model

In case we want to derive the reduced phase space with respect to the Hamiltonian as well as the spatial diffeomorphism constraints, we need to introduce four reference fields in total. Coming from the model discussed in the last section, a natural candidate for such a model is the four KG scalar fields model described by the following action:

$$S\left[g,\varphi^0,\varphi^j\right]=\frac{1}{\kappa}\int \mathrm{d}^4X\sqrt{\det(g)}R+\frac{1}{2}\int \mathrm{d}^4X g^{\mu\nu}\delta_{IJ}\varphi^I_{,\mu}\varphi^J_{,\nu}$$

with $I,J=0,1,2,3$ and $\varphi^I=(\varphi^0,\varphi^j)$ with $j=1,2,3$ and φ^j are the three reference fields for the spatial coordinates. The detailed constraint analysis of this model is described in [5] and we end up with the following spatial diffeomorphism and Hamiltonian constraints

$$c=c^{\mathrm{geo}}+\frac{1}{2}\frac{\delta^{IJ}\pi_I\pi_J}{\sqrt{q}}+\frac{1}{2}\sqrt{q}\delta_{IJ}q^{ab}\varphi^I_{,a}\varphi^J_{,b}\,,\qquad c_a=c^{\mathrm{geo}}_a+\pi_J\varphi^J_{,a}\,,$$

where π_J are the canonical conjugate momenta to φ^J, q_{ab} is again understood as a function of (A,E) and $c^{\mathrm{geo}},c^{\mathrm{geo}}_a$ denote the geometric part of the Hamiltonian and spatial diffeomorphism constraint, respectively. As shown in detail in [5], a set of abelianized constraints can be obtained by solving c for π_0 and c_a for π_j leading to

$$\begin{aligned}\tilde{c} &= \pi_0-h\left(A,E,\varphi^0,\varphi^j\right)\,, &\qquad h&=-\frac{b}{2a}\pm\sqrt{\left(\frac{b}{2a}\right)^2-\frac{c}{a}}\,,\\ \tilde{c}_j &= \pi_j-h_j\left(A,E,\varphi^0,\varphi^j\right)\,, &\qquad h_j&=\varphi^b_j\left(c^{\mathrm{geo}}_b+h\varphi^0_{,b}\right)\end{aligned}$$

with

$$\begin{aligned}a &:= \left(1+\delta^{jk}\varphi^a_j\varphi^b_k\varphi^0_{,a}\varphi^0_{,b}\right)\,, \qquad b:=\delta^{jk}\varphi^a_j\varphi^b_k\left(c^{\mathrm{geo}}_a\varphi^0_{,b}+c^{\mathrm{geo}}_b\varphi^0_{,a}\right)\,,\\ c &:= q\delta_{jk}q^{ab}\varphi^j_{,a}\varphi^k_{,b}+\delta^{jk}\varphi^a_j\varphi^b_k c^{\mathrm{geo}}_a c^{\mathrm{geo}}_b+2\sqrt{q}c^{\mathrm{geo}}\,.\end{aligned}$$

Now, if we construct observables $O_{A,(\varphi^0,\varphi^j)}(\sigma^j,\tau),O_{E,(\varphi^0,\varphi^j)}(\sigma^j,\tau)$ with respect to $\tilde{c}$ and $\tilde{c}_j$ using the reference fields φ^0,φ^j in this model, we obtain the following form of the physical Hamiltonian [5]:

$$H_{\mathrm{phys}}=\int_{\mathcal{S}}\mathrm{d}^3\sigma\sqrt{-\left(2\sqrt{Q}\,C^{\mathrm{geo}}+QQ^{jk}\delta_{jk}+C^{\mathrm{geo}}_jC^{\mathrm{geo}}_k\delta^{jk}\right)}\,.$$

Here, $\mathcal{S}$ is the so-called scalar field manifold coordinatized by the values σ^j that the scalar fields φ^j can take. The next step in the reduced quantization

program is to look for representations of the observable algebra, that in the four KG fields model has again the standard canonical form [5]

$$\left\{O_{A,(\varphi^0,\varphi^j)}(\tau,\sigma), O_{E,(\varphi^0,\varphi^j)}\left(\tau,\sigma'\right)\right\} = \delta^{(3)}\left(\sigma,\sigma'\right) .$$

Here, a natural choice is the AS representation by going over to the holonomy-flux algebra. However, we are only interested in those representations for which the physical Hamiltonian can be promoted to a well-defined operator, as otherwise the quantum dynamics cannot be formulated. If we choose this representation and look at the explicit form of the physical Hamiltonian density, the following problem occurs: Because the unitary operators associated with the spatial diffeomorphisms are not implemented weakly continuously in the AS representation and hence the infinitesimal generators $\hat{C}_j^{\mathrm{geo}}$ do not exist. However, $H(\sigma)$ involves a term of the form $\delta^{jk}C_j^{\mathrm{geo}}C_k^{\mathrm{geo}}$ and hence we realize that $H(\sigma)$ cannot be quantized using the AS representation. This issue does not arise in the other existing models because there the spatial diffeomorphism constraints occurs in the combination $Q^{jk}C_j^{\mathrm{geo}}C_k^{\mathrm{geo}}$ and this can indeed be quantized using LQG techniques. We conclude that in the case of the one and the four KG scalar field(s) model, Dirac and reduced quantization yield to very different results. In the first case, the physical Hilbert space can be obtained whereas in the latter case a quantization of the classical reduced theory is not even possible with standard LQG techniques. In the next section, we introduce a generalization of the four KG scalar fields model that cures this problem and show that in a certain limit Dirac quantization and reduced quantization lead to the same quantum dynamics.

4.1. Generalization of the four Klein–Gordon scalar field model

We consider the following generalization of the four scalar field model introduced in [5]

$$\begin{aligned} S[g,\varphi^0,\varphi^j,M_{jj}] \;=\; & \frac{1}{\kappa}\int \mathrm{d}^4X\sqrt{\det(g)}R \\ & +\frac{1}{2}\int \mathrm{d}^4X g^{\mu\nu}\left(\varphi^0_{,\mu}\varphi^0_{,\nu}+\sum_{j=1}^{3}M_{jj}\varphi^j_{,\mu}\varphi^j_{,\nu}\right) , \end{aligned}$$

as before, the reference fields are $\varphi^I = (\varphi^0,\varphi^j)$ with $j = 1,2,3$. The Kronecker delta δ_{jk} in the spatial part has been replaced by a more general diagonal matrix M_{jk} that involves 3 additional dynamical degrees of freedom (dof) sitting in M_{jj}. Hence, in addition to GR, we have 7 more scalar field dof. Likewise to the models in the seminal papers [11, 14], the system

with these more than four additional dof exhibits second class constraints. The reduction with respect to these second class constraints yields a phase space with four additional scalar fields and first class constraints only. Here, the motivation of the generalization of our model follows exactly this idea. As discussed in detailed form in [5], the generalized model has 6 second class constraints which reduce the three additional dof sitting in M_{jj}. After reduction with respect to the second class constraints, we end up next to the Gauss constraint with the following first class constraints:

$$\begin{aligned} c_a^{\rm tot} &= c_a^{\rm geo} + \pi_0 \varphi^0_{,a} + \pi_j \varphi^j_{,a}\,, \\ c^{\rm tot} &= c^{\rm geo} + \frac{\pi_0^2}{2\sqrt{q}} + \frac{1}{2}\sqrt{q} q^{ab} \varphi^0_{,a} \varphi^0_{,b} + \sum_{j=1}^{3} \varphi^a_j \left(c_a^{\rm geo} + \pi_0 \varphi^0_{,a}\right) \sqrt{q^{bc} \varphi^j_{,b} \varphi^j_{,c}}\,. \end{aligned}$$

Likewise to the former model, we can solve $c^{\rm tot}$ for π_0 and $c_a^{\rm tot}$ for π_j and consider the equivalent set of constraints

$$\begin{aligned} \tilde{c}^{\rm tot} &:= \pi_0 - h\left(q_{ab}, p^{ab}, \varphi^0, \varphi^j\right)\,, \qquad h = -\frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 - c}\,, \\ \tilde{c}_j^{\rm tot} &:= \pi_j - h_j\left(q_{ab}, p^{ab}, \varphi^0, \varphi^j\right)\,, \qquad h_j = \varphi^a_j \left(c_a^{\rm geo} + \pi_0 \varphi^0_{,a}\right)\,, \end{aligned}$$

now with

$$\begin{aligned} b &:= 2\sqrt{q} \sum_{j=1}^{3} \varphi^0_{,a} \varphi^a_j \sqrt{q^{cd} \varphi^j_{,c} \varphi^j_{,d}}\,, \\ c &:= q q^{ab} \varphi^0_{,a} \varphi^0_{,b} + 2\sqrt{q} \sum_{j=1}^{3} \varphi^a_j c_a^{\rm geo} \sqrt{q^{bc} \varphi^j_{,b} \varphi^j_{,c}} + 2\sqrt{q} c^{\rm geo}\,. \end{aligned}$$

After the construction of observables with respect to $\tilde{c}^{\rm tot}, \tilde{c}_j^{\rm tot}$, as shown in [5], their algebra has the standard canonical form given by [5]

$$\left\{O_{A,(\varphi^0,\varphi^j)}(\tau,\sigma), O_{E,(\varphi^0,\varphi^j)}\left(\tau,\sigma'\right)\right\} = \delta^{(3)}\left(\sigma,\sigma'\right)\,.$$

The dynamics of these observables is encoded in the first order Hamiltonian equations

$$\begin{aligned} \frac{O_{A,(\varphi^0,\varphi^j)}}{{\rm d}\tau}(\sigma,\tau) &= \{O_{A,(\varphi^0,\varphi^j)}(\sigma,\tau), H_{\rm phys}\}\,, \\ \frac{O_{E,(\varphi^0,\varphi^j)}}{{\rm d}\tau}(\sigma,\tau) &= \{O_{E,(\varphi^0,\varphi^j)}(\sigma,\tau), H_{\rm phys}\} \end{aligned}$$

with a physical Hamiltonian of the form of

$$H_{\text{phys}} = \int_{\mathcal{S}} \mathrm{d}^3\sigma \sqrt{-2\sqrt{Q}\, C^{\text{geo}} - 2\sqrt{Q}\sum_{j=1}^{3}\sqrt{Q^{jj}C_j^{\text{geo}}C_j^{\text{geo}}}}(\sigma)\,. \tag{4.1}$$

We realize that in H_{phys} of this generalized model, the spatial diffeomorphism constraints occur only in the combination $Q^{jj}C_j^{\text{geo}}C_j^{\text{geo}}$ which is a specific case of the combination $Q^{jk}C_j^{\text{geo}}C_k^{\text{geo}}$. For the latter, it has been shown in [1] that a well-defined operator exists in the AS representation. The main reason for this to be the case is that here, the C_j^{geo}s are contracted with the inverse metric components and not just with a Kronecker delta as it was the case in the four KG scalar field model without the generalization. Thus, in this model, the reduced phase space quantization can be performed. Moreover, we can choose the AS representation for $\mathcal{H}_{\text{phys}}$ and use the techniques introduced in [1,15] to formulate the quantum dynamics as discussed in [5].

5. Discussion and conclusions

In this letter, we summarized the main features of the scalar fields models introduced in [3] and [5]. The first one involves only one KG reference scalar field, whereas the latter one involves four KG reference fields. As a consequence, the spatial diffeomorphism constraints are solved via Dirac quantization in [3] and via reduced quantization in [5]. Both models can be understood as natural generalizations of the APS model in loop quantum cosmology to full LQG. Surprisingly, it turns out that the naive generalization of the APS model by coupling four KG fields to gravity leads to a reduced model whose classical dynamics cannot be quantized using the Ashtekar–Lewandowski representation of LQG. However, to circumvent this problem, a slight generalization of this model is introduced that considers three additional scalar fields along the lines of the seminal models in [12,14]. This yields to a model with a reduced phase space and reduced dynamics that can be quantized using standard techniques developed for LQG. We realize that the two physical Hamiltonians in (3.2) and (4.1) differ. However, the second term under the square root form in (4.1) can be related to the momentum density of the scalar fields φ^j and this can be understood as a fingerprint of the dynamically coupled observer onto the system that is absent in the case of Dirac quantization. In the limit, where this is assumed to vanish, the two physical Hamiltonians agree.

A.O. thanks the Heinrich-Böll-Foundation for financial support. K.G. thanks the organizers of the 3rd Conference of the Polish Society on Relativity for their kind invitation.

REFERENCES

[1] K. Giesel, T. Thiemann, *Class. Quantum Grav.* **27**, 175009 (2010).
[2] V. Husain, T. Pawłowski, *Phys. Rev. Lett.* **108**, 141301 (2012).
[3] M. Domagała, K. Giesel, W. Kamiński, J. Lewandowski, *Phys. Rev. D* **82**, 104038 (2010).
[4] K. Giesel, T. Thiemann, *Class. Quantum Grav.* **32**, 135015 (2015).
[5] K. Giesel, A. Oelmann, `arXiv:1610.07422 [gr-qc]`.
[6] K. Giesel, H. Sahlmann, *PoS* **QGQGS 2011**, 002 (2011).
[7] C. Rovelli, *Class. Quantum Grav.* **8**, 297 (1991); **8**, 317 (1991); *Phys. Rev. D* **43**, 442 (1991); **42**, 2638 (1990).
[8] B. Dittrich, *Gen. Relativ. Gravitation* **39**, 1891 (2009); *Class. Quantum Grav.* **23**, 6155 (2006).
[9] T. Thiemann, *Class. Quantum Grav.* **23**, 1163 (2006).
[10] K. Giesel, *Int. J. Mod. Phys. A* **23**, 1190 (2008).
[11] K.V. Kuchař, J.D. Romano, *Phys. Rev. D* **51**, 5579 (1995).
[12] J.D. Brown, K.V. Kuchař, *Phys. Rev. D* **51**, 5600 (1995).
[13] A. Ashtekar, T. Pawlowski, P. Singh, *Phys. Rev. D* **74**, 084003 (2006).
[14] K.V. Kuchař, C.G. Torre, *Phys. Rev. D* **44**, 3116 (1991).
[15] T. Thiemann, *Phys. Lett. B* **380**, 257 (1996).