LOOP QUANTUM GRAVITY WITH SCALAR FIELD AS A PHYSICAL TIME VARIABLE*

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By using a scalar field as a relational time variable for the dynamics of the gravitational field, we construct mathematically complete and well-defined models of loop quantum gravity, in which the dynamics of quantum geometry is explicitly computable, at least in certain simple examples.

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1. Introduction

Loop quantum gravity [1–3] is based on a canonical quantization of general relativity in the Ashtekar formulation, in which the basic variables are the connection A_a^i and the densitized triad E_i^a of the (inverse) spatial metric q^{ab}. In this formulation, general relativity is encoded in the Gauss, diffeomorphism, and Hamiltonian constraints, arising from gauge invariance under local rotations of the triad, and under diffeomorphisms tangent and orthogonal to the spatial surfaces of the 3+1 decomposition of spacetime.

In the quantum theory, implementation of the Gauss and diffeomorphism constraints is straightforward. Their solution space is spanned by the so-called spin network states, which give a kinematical description of a quantized, discrete spatial geometry. However, describing the dynamics of these states through quantizing the Hamiltonian constraint and looking for its solutions has proven to be technically extremely challenging. An alternative approach to the problem of dynamics in loop quantum gravity is provided by the so-called method of deparametrization, in which we consider gravity coupled to a matter field, and use the matter field as a physical, relational time variable, with respect to which the evolution of the quantum state of the gravitational field is described.

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2. The physical Hamiltonian in deparametrized LQG

In practice, the most useful choice for the reference matter field is a scalar field: either a free Klein–Gordon field [4–6] or an irrotational dust field [7,8]. The physical Hamiltonian, which generates the dynamics of a spin network state $|\Psi\rangle$ through the Schrödinger equation $i\frac{\mathrm{d}}{\mathrm{d}\phi}|\Psi\rangle = \hat{H}_{\rm phys}|\Psi\rangle$, is given respectively in the two cases by

$$\hat{H}_{\rm phys} = \int \mathrm{d}^3x\,\widehat{\sqrt{-2\sqrt{q}C}} \qquad \text{or} \qquad \hat{H}_{\rm phys} = \int \mathrm{d}^3x\,\widehat{C}\,, \tag{1}$$

where C denotes the gravitational Hamiltonian constraint.

In order to give an explicit construction [6] of the formal objects in (1), thereby completing the definition of the dynamics of the theory, it is useful to start with the following classical identity. The gravitational Hamiltonian constraint is usually expressed as

$$C = \frac{\epsilon_{ijk}E^a_iE^b_jF^k_{ab}}{\sqrt{q}} + \frac{2\left(1+\beta^2\right)}{\sqrt{q}}E^a_iE^b_jK^i_{[a}K^j_{b]}\,, \tag{2}$$

where F^i_{ab} is the curvature of the Ashtekar connection, β is the Barbero–Immirzi parameter, and $K^i_a = K_{ab}E^b_i/\sqrt{q}$, with K_{ab} the extrinsic curvature of the spatial surface. However, modulo terms proportional to the Gauss constraint, expression (2) is identically equal to

$$C = \frac{1}{\beta^2}\frac{\epsilon_{ijk}E^a_iE^b_jF^k_{ab}}{\sqrt{q}} + \frac{1+\beta^2}{\beta^2}\sqrt{q}\,{}^{(3)}R\,, \tag{3}$$

where ${}^{(3)}R$ is the Ricci scalar of the spatial surface.

The advantage of this rewriting is that an operator corresponding to the second term of (3) can be constructed [9] based on Regge's formula, which approximates the integral $\int \mathrm{d}^3x\,\sqrt{q}\,{}^{(3)}R$ in terms of the hinge lengths and deficit angles of a cellular decomposition of the spatial surface. These can be quantized using the length and angle operators available in loop quantum gravity, and the resulting operator is remarkably simple in comparison to the operator obtained earlier by quantizing the second term of (2).

To quantize the first term of (3), the curvature of the Ashtekar connection must be regularized in terms of the holonomy of the connection around a small loop. It is then important to specify this loop in such a way that the resulting operator can be made symmetric (in order to eventually obtain a self-adjoint physical Hamiltonian). In particular, the loop assignment introduced by Thiemann in order to define the Hamiltonian constraint operator [10] is not suitable for constructing a symmetric operator. In [6], we provide an improved prescription for the loop, which is sufficient to ensure that a symmetric Hamiltonian is obtained.

3. Computing the time evolution of spin network states

With an explicit construction of the physical Hamiltonian operator completed, it becomes possible to study the dynamics of the resulting theory through concrete calculations. An exact evaluation of the time evolution of a given spin network state seems a very difficult task to achieve, as it would require one to diagonalize the rather complicated physical Hamiltonian (or at least the restriction of the operator to a certain subspace of the full Hilbert space of the theory). However, calculations making use of various approximations can still be carried out.

For large values of the Barbero–Immirzi parameter β, the relatively complicated first term in (3) can be considered as a perturbation over the significantly simpler second term, whose eigenstates and eigenvalues can be computed at least numerically. This enables one to develop an approximate spectral decomposition of the full physical Hamiltonian based on the corresponding decomposition of the unperturbed Hamiltonian, using standard time-independent perturbation theory of quantum mechanics. This approach is particularly relevant to the case of the free Klein–Gordon field as the physical time variable, since in that case, the physical Hamiltonian involves a square root, and a spectral decomposition of the operator under the square root is required in order to give a concrete definition of the Hamiltonian. For a more detailed discussion of this method, including (numerical) examples of using it to compute the dynamics of simple states of quantum geometry, we refer the reader to our recent article [11].

In the case of the dust field, the physical Hamiltonian involves no square root, and its action is computable regardless of the value of the Barbero–Immirzi parameter. Time evolution over a sufficiently short interval of time can then be calculated simply by expanding the quantity of interest in powers of the time variable. For example, the time-dependent expectation value of a geometrical observable such as the volume operator would have the form of $\langle V(t)\rangle = v_0 + v_1 t + v_2 t^2 + \ldots$, with the coefficients given by expectation values of repeated commutators of the volume with the Hamiltonian in the initial state. The first few coefficients of such an expansion can be computed numerically, at least for simple enough initial states. Concrete examples are again given in [11].

4. Summary and outlook

The problem of dynamics in loop quantum gravity can be treated by using a scalar field as a relational time variable for the evolution of the quantized gravitational field. By giving an explicit construction of the physical Hamiltonian that governs the quantum dynamics of the gravitational field, we complete the definition of a class of mathematically complete and well-defined models of loop quantum gravity, and make it possible to investigate the dynamics of such models through concrete calculations.

To gain a deeper understanding of the physical content of these models, one should study the dynamics of states having a clear and satisfactory physical interpretation. While the well-known heat kernel coherent states have good kinematical peakedness properties with respect to certain loop quantum gravity operators [12, 13], they are based on a fixed graph, and hence it seems unlikely that they would be dynamically coherent under a graph-changing Hamiltonian. The problem of developing coherent states compatible with the dynamics of our models is at the moment fully open.

From a practical point of view, it would undoubtedly be useful to further develop approximation methods for performing calculations within our models — preferably methods not relying heavily on numerical computations. A recent attempt in this direction has been made in [14], where we use angular momentum coherent states to develop a scheme that can be used to analyze the matrix elements of our Hamiltonian.

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