KUNDT'S METRICS WITH SPECIAL PROPERTIES*

Maciej Kruk, Jerzy Lewandowski, Adam Szereszewski

Institute of Theoretical Physics, Faculty of Physics, University of Warsaw
Pasteura 5, 02-093 Warszawa, Poland

The structure of vacuum Einstein's field equations for n-dimensional Kundt's metrics is described. Using the Bianchi identity, it is shown that if a part of these equations is satisfied, then the remaining can be reduced to linear system. This observation can be used to generate new solutions from old ones. Starting from the known solution, we construct the metric of type II. We also resolve the discrepancy in general form of type D vacuum Kundt's spacetime which has been arisen in the literature.

1. Introduction

From the very beginning of General Relativity searching for exact solutions played a crucial role. In the mid 20$^{\text{th}}$ century, it became clear that certain geometrical properties of spacetime geometry might lead to new interesting solutions of Einstein's field equations. Kundt's class [1] is one of the most prominent example of such spacetimes. It includes not only the pp-waves and plane wave solutions [2] but also spacetimes foliated by non-expanding horizons [3–5] and so-called near horizon geometries [6,7].

2. Kundt's class

Spacetime belongs to Kundt's class if it admits a null geodesic congruence generated by a vector field ℓ which is shear-free, twist-free and expansion-free. In order to construct n-dimensional Kundt's metric, we start with n-dimensional manifold M equipped with metric g and local coordinates (x^α). If ℓ is the vector field with properties described above, then two coordinates u and v can be naturally introduced by $\ell = \partial_v$, $-\ell_\alpha \mathrm{d}x^\alpha = \mathrm{d}u$,

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where v is chosen to be an affine parameter. Then the n-dimensional metric of Kundt's class has the following form [2]:

$$g = g_{AB}\mathrm{d}x^A\mathrm{d}x^B - 2\mathrm{d}u\left(\mathrm{d}v + W_A\mathrm{d}x^A + H\mathrm{d}u\right), \qquad (x^\alpha) = \left(x^A, v, u\right), \quad (1)$$

where $g_{AB,v} = 0$ and $W_A = W_A(x^\alpha)$, $H = H(x^\alpha)$. This form of line element (1) determines the coordinates modulo the following transformations

$$
\begin{array}{llll}
(i) & x^A = \bar{x}^A, & u = U(\bar{u}), & v = \dfrac{\bar{v}}{U'(\bar{u})}, & (2) \\[2ex]
(ii) & x^A = \bar{x}^A, & u = \bar{u}, & v = \bar{v} + f\left(\bar{x}^A, \bar{u}\right), & (3) \\[1ex]
(iii) & x^A = f^A\left(\bar{x}^A, \bar{u}\right), & u = \bar{u}, & v = \bar{v}, & (4)
\end{array}
$$

supplemented by the proper transformation of functions g_{AB}, W_A and H.

2.1. Ricci tensor and Bianchi identity

In the present note, we shall consider Kundt's spacetimes satisfying vacuum Einstein's equations $R_{\alpha\beta} = 0$. Two equations $R_{Av} = R_{uv} = 0$ can easily be integrated yielding

$$W_A = 2\omega_A v + \phi_A, \qquad H = H_2 v^2 + H_1 v + H_0, \tag{5}$$

where

$$H_2 = \tfrac{1}{2}\left(D_A\omega^A - 2\omega^A\omega_A\right) \tag{6}$$

and new functions ω_A, ϕ_A, H_1, H_0 do not depend on v. Hence, the dependence of all the metric coefficients $g_{\alpha\beta}$ on v is known. After the substitution (5), the remaining coefficients of the Ricci tensor reduce to

$$R_{AB} = D_A\omega_B + D_B\omega_A - 2\omega_A\omega_B + \hat{\mathsf{R}}^{(n-2)}_{AB} \tag{7}$$

and $R_{Au} = R^{(1)}_{Au}v + R^{(0)}_{Au}$, $R_{uu} = R^{(2)}_{uu}v^2 + R^{(1)}_{uu}v + R^{(0)}_{uu}$, where quantities $R^{(1)}_{Au}$, $R^{(0)}_{Au}$, $R^{(2)}_{uu}$, $R^{(1)}_{uu}$, $R^{(0)}_{uu}$ are independent of v. It turns out that, as a consequence of Bianchi identity $\nabla_\alpha G^\alpha{}_\beta = 0$, some of the equations are automatically satisfied. As a result, Einstein's field equations reduce to

$$D_A\omega_B + D_B\omega_A - 2\omega_A\omega_B + \hat{\mathsf{R}}^{(n-2)}_{AB} = 0 \tag{8}$$

and $R^{(0)}_{Au} = R^{(0)}_{uu} = 0$.

2.2. Method of solving

The Bianchi identity discussed in the previous section leads to the following method of solving vacuum Einstein's equations. Starting from a known solution (g_{AB}, ω_A) of (8), one can obtain a full solution of vacuum Einstein's equations by:

— solving linear system of $n-2$ equations for functions ϕ_A and H_1

$$D^B D_{[B}\phi_{A]} + \left(\omega^B D_A - \omega_A D^B\right)\phi_B + \left(D_A\omega^B - 2D^B\omega_A + 2\omega_A\omega^B\right)\phi_B$$
$$+H_{1,A} + g^{BC}D_{[B}g_{A]C,u} - \frac{1}{2}g^{BC}g_{BC,u}\omega_A - \omega_{A,u} = 0\,, \tag{9}$$

— and then finding H_0 which satisfies the following linear equation:

$$D^A D_A H_0 + 2D_A\left(\omega^A H_0\right) - 2\phi^A H_{1,A} - S^A_A H_1 + \phi^A\phi_A D_B\omega^B$$
$$-D^A\phi_{A,u} + 2\phi^A\omega_{A,u} - \frac{1}{2}g^{AB}g_{AB,uu} + L_{AB}L^{AB} + 4\phi^A\omega^B D_{[A}\phi_{B]}$$
$$-2\phi^A\phi^B\omega_A\omega_B = 0\,,$$

where $S_{AB} = D_{(A}W_{B)} + \frac{1}{2}g_{AB,u}$, $L_{AB} = D_{[A}W_{B]} + \frac{1}{2}g_{AB,u}$.

3. Vacuum Kundt's spacetimes

In this section, we shall apply the method of solving described above. We start with the following quantities $(k, l, m = \text{const})$

$$g_{AB}\mathrm{d}x^A\mathrm{d}x^B = P^2\mathrm{d}z^2 + P^{-2}\mathrm{d}y^2\,, \qquad P^2 = \frac{z^2 + l^2}{k\left(z^2 - l^2\right) + 2mz}\,, \tag{10}$$

$$\omega_1 = -z\left(z^2 + l^2\right)^{-1}\,, \qquad \omega_2 = l\left(z^2 + l^2\right)^{-1}P^{-2}\,, \tag{11}$$

which satisfy (8) for $n = 4$. According to (6), we obtain

$$H_2 = -\left(z^2 + l^2\right)^{-1}\left[k/2 + 2l^2\left(z^2 + l^2\right)^{-1}P^{-2}\right]\,. \tag{12}$$

Assuming $\phi_A = 0$, the method of solving leads to $H_1 = 0$ (see (9)) and a second order linear partial differential equation for $H_0(y, z, u)$. If H_0 depends only on z and u, then the general solution reads

$$H_0 = \left(z^2 + l^2\right)\left[A(u) + B(u)\text{arctanh}\left((kz + m)/\sqrt{k^2l^2 + m^2}\right)\right]\,. \tag{13}$$

It can be checked that the solution presented above is of type II if $B \neq 0$, whereas $B = 0$ case leads to type D metric.

According to [2], all vacuum solutions of type D in Kundt's class are completely known and they are given by (1) with $W_A = 2\omega_A v$, $H = H_2 v^2$ and (10)–(12). On the other hand, type D solutions presented in [8] contain more general H, namely[1]

$$H = H_2 v^2 + \epsilon_0 \left(z^2 + l^2\right)/2 \,, \qquad \epsilon_0 = \text{const} \tag{14}$$

The method of solving described above leads to solutions of type D with

$$H = H_2 v^2 + A(u) \left(z^2 + l^2\right) \,, \tag{15}$$

which seems to be even more general than (14).

This discrepancy is resolved by the fact that function $H_0 = A(u)(z^2 + l^2)$ may be removed from metric by the following coordinate transformation $y \mapsto y + h(u)$, $u \mapsto U(u)$, $v \mapsto v/U'(u) + f(z,u)$, where

$$f(z,u) = \left(z^2 + l^2\right) h' l^{-1} Q^2/2 \,, \qquad h(u) = 4l k^{-1} \log |Q| \tag{16}$$

and $Q(u)$ fulfils $2Q'' + kA(u)Q^{-3} = 0$, $U' = Q^{-2}$.

To summarise, we have shown that the method of solving vacuum Einstein's equation for Kundt's class leads to type II solution characterised by function H_0 given in (13). Assuming $B(u) = 0$ in (13) reduces Kundt's metric to solution of type D in the Petrov classification.

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[1] It is noted that the solution in [8] is non-vacuum and includes electric and magnetic charge as well as cosmological constant.