

SOLUTION WITH SEPARABLE VARIABLES FOR NULL ONE-WAY MAXWELL FIELD IN KERR SPACE-TIME*

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We have found in analytic form an exact solution with separable variables for null one-way Maxwell field $\varphi_{AB} = \varphi_2 o_A o_B$ on the Kerr space-time background and have investigated some of its properties. Solution describes outgoing waves when $r > r_{\text{cr.1}} > r_+$, but for some Maxwell field parameters, this solution describes ingoing, standing and outgoing waves on defined intervals in the region of $r > r_+$.

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1. Introduction

The problem of evolution of test fields of spin s in the Kerr space-time (KST) background had been substantially solved [1], but still contains mathematical difficulties of theoretical and computational character: separation of variables (SOV) determines nonlinear eigenvalue problem; numerical and approximate methods that are used are faced with computational difficulties [2].

Consideration of algebraically special (degenerate) Maxwell fields gives an exact solution in analytic form as an arbitrary function of two variables, which are complex integrals of system's equations in Minkowski space-time [3] and KST [4, 5]. A solution as an arbitrary function does not reveal clear physical properties, and for looking of its applicability, it is necessary to build a solution with separated variables. This article is a continuation of our previous work [4], the most of introduction can be found there. Our purpose is to find a solution with separated variables for null one-way (NOW) Maxwell field

$$\varphi_{AB} = \varphi_2 o_A o_B \quad (1)$$

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(it is one of two algebraically special fields) in KST, which in the Boyer–Lindquist coordinates is given by the line element

$$ds^2 = \left(1 - \frac{2Mr}{\Sigma}\right) dt^2 + \frac{4Mra \sin^2 \theta}{\Sigma} dt d\phi - \frac{\Sigma}{\Delta} dr^2 - \Sigma d\theta^2 - \left(r^2 + a^2 + \frac{2Mra^2 \sin^2 \theta}{\Sigma}\right) \sin^2 \theta d\phi^2, \quad (2)$$

where $\Sigma = r^2 + a^2 \cos^2 \theta$, $\Delta = r^2 - 2Mr + a^2 = (r - r_+)(r - r_-)$, $r_{\pm} = M \pm \sqrt{M^2 - a^2}$; and to investigate its properties. Though a solution of this type has not been considered in Teukolsky's approach, we will show that it has some similar properties in the regions that do not include singular points what confirms advisability of further study.

By choosing a Newman–Penrose null tetrad as the Kinnersley tetrad, we write down a decoupled system of Maxwell equations in spinor form for NOW field. In the next section, we will apply a method of separation of variables (SOV) to the first order system of equations and will find its solution. Equations are considered in geometrized units $c = G = 1$.

2. Separated solution of Maxwell equations in Kerr space-time for null one-way field

Let us consider the system of Maxwell equations for NOW field

$$\begin{cases} \frac{r^2+a^2}{\Delta} \frac{\partial \varphi_2}{\partial t} + \frac{\partial \varphi_2}{\partial r} + \frac{a}{\Delta} \frac{\partial \varphi_2}{\partial \phi} + \frac{1}{r-ia \cos \theta} \varphi_2 = 0, \\ ia \sin \theta \frac{\partial \varphi_2}{\partial t} + \frac{\partial \varphi_2}{\partial \theta} + \frac{i}{\sin \theta} \frac{\partial \varphi_2}{\partial \phi} + \left(\operatorname{ctg} \theta + \frac{ia \sin \theta}{r-ia \cos \theta} \right) \varphi_2 = 0. \end{cases} \quad (3)$$

As in general case of SOV [1], separability in t and ϕ follows directly from stationarity and axisymmetry of the Kerr metric, and we have $\varphi_2(t, r, \theta, \phi) = e^{i\omega t + im\phi} \hat{\varphi}_2(r, \theta)$. We consider $\omega \in \mathbb{R}$ and $m \in \mathbb{Z}$.

After rewriting equations for function $\psi(r, \theta) = (r - ia \cos \theta) \hat{\varphi}_2(r, \theta)$, SOV in the form of $\psi(r, \theta) = R(r)S(\theta)$ is provided automatically, and, in our special case, additional separation constants do not emerge. Thus, we obtain a system of two ODEs

$$R'(r) + \left(i\omega \frac{r^2 + a^2}{r^2 - 2Mr + a^2} + im \frac{a}{r^2 - 2Mr + a^2} \right) R(r) = 0, \quad (4a)$$

$$S'(\theta) + \left(\operatorname{ctg} \theta - a\omega \sin \theta - m \frac{1}{\sin \theta} \right) S(\theta) = 0. \quad (4b)$$

Points $\theta = 0$, $\theta = \pi$ for equation (4b) are regular singular points. The solution of equation (4b) is

$$S(\theta) = C_2 \frac{(1 - \cos \theta)^m}{\sin^{m+1} \theta} e^{-a\omega \cos \theta}, \quad (5)$$

$C_2 \in \mathbb{C}$. For $m = 0, -1, -2, \dots$, $S(\theta)$ becomes infinite at point $\theta = 0$ and for $m = 0, 1, 2, \dots$ — at point $\theta = \pi$. We expect that $S(\theta)$ and full solution will have physical meaning beyond these points.

It is not surprising that radial equation (4a), as well as Teukolsky radial equation (TRE), have two regular singular points at the horizon radii r_+ and r_- . It has also an irregular singular point at infinity.

The solution of equation (4a) is

$$R(r) = C_1 e^{-i\omega r_*}, \quad (6)$$

where $r_* = r + M \ln |\Delta| + \frac{2\omega M^2 + ma}{2\omega \sqrt{M^2 - a^2}} \ln \left| \frac{r - r_+}{r - r_-} \right|$, $C_1 \in \mathbb{C}$.

By comparing with the general case, where solution is built as series of $(r - r_+)/(r - r_-)$ [6], radial NOW solution (6) is an elementary function of $(r - r_+)/(r - r_-)$. Asymptotically (at positive infinity), the solution of TRE [1] has the form of (6).

Further, we find critical points of $R(r)$, which are r_- , r_+ , $r_{\text{cr.1,2}} = \pm \sqrt{-am/\omega - a^2}$. At these points, the increasing–decreasing character of r_* changes, what means that the wave changes its outgoing–ingoing behavior. But in all the cases at $r \rightarrow \infty$, function $r_*(r)$ is increasing that means the wave is outgoing. That is why we call the solution “outgoing”.

Critical points $r_{\text{cr.1,2}}$ for some values of m exist only in the special case $-m > a\omega$, and their localization depends on values of parameters a , M of the Kerr metric and Maxwell field characteristics m and ω . Let us describe the properties of function $R(r)$ in neighborhood of point $r_{\text{cr.1}}$. When condition $r_{\text{cr.1}} > r_+$ is fulfilled, we obtain the wave, which is ingoing for $r_+ < r < r_{\text{cr.1}}$, standing on surface $r = r_{\text{cr.1}}$, and outgoing for $r > r_{\text{cr.1}}$. Superradiation condition $m\omega_+/\omega > 1$, established by Teukolsky [1] and Starobinski and Churilov [7], is equivalent with the condition of existence of the point $r_{\text{cr.1}}$ outside the horizon: $r_{\text{cr.1}} > r_+$.

For example, we plot the function $\text{Re}(R(r))$. The following parameters $C = 1$, $a = 0.8$, $M = 1$, $m = 6$, $\omega = -1$ and $r_{\text{cr.1}} > r_+$ are chosen (Fig. 1). For $r'_*(r) \cdot r > 0$, the wave $e^{i\omega(t-r_*)}$ is outgoing, and for $r'_*(r) \cdot r < 0$ — ingoing.

Finally, the solution of system (3) is

$$\varphi_2(t, r, \theta, \phi) = C e^{i\omega(t-r_*) + im\phi} \frac{e^{-a\omega \cos \theta} (1 - \cos \theta)^m}{(r - ia \cos \theta) \sin^{m+1} \theta}. \quad (7)$$

Teukolsky wrote down an expression for total energy flux per unit solid angle for outgoing waves at infinity ((5.13) in [1]). Having solution (7), we calculate T_{tr} component of energy momentum tensor for NOW Maxwell field

and flux for all values of r and $\theta \neq 0, \theta \neq \pi$

$$\frac{d^2 E}{dt d\Omega} = r^2 T_{tr} = |C|^2 \frac{r^2 (1 - \cos \theta)^{2m}}{\Delta \sin^{2(m+1)} \theta} e^{-2a\omega \cos \theta}. \quad (8)$$

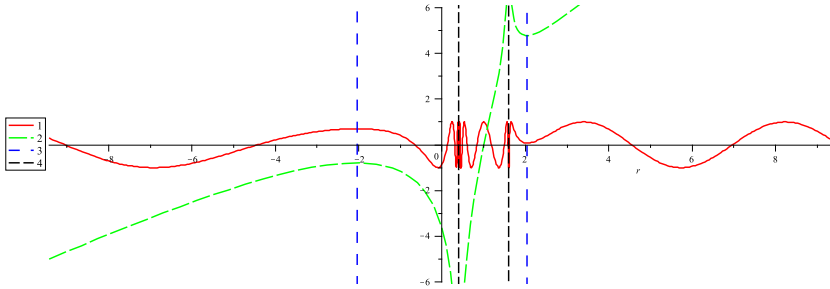


Fig. 1. Graph of the real part of $R(r)$ with $C = 1$, $a = 0.8$, $M = 1$, $m = 6$, $\omega = -1$ ($r_{\text{cr},1} > r_+$). 1 — $\text{Re}(R(r))$; 2 — $r_*(r)$; 3 — critical points $r_{\text{cr},1}$, $r_{\text{cr},2}$; 4 — outer and inner horizons r_+ , r_- .

3. Conclusions

For null one-way test Maxwell field in KST, we have considered the first order system of PDEs and have obtained a solution in closed form by method of separation variables. Some property of this solution joins NOW Maxwell field angular momentum, frequency and Kerr black hole angular momentum in superradiant condition. Electromagnetic waves with parameters $r_{\text{cr},1} > r_+$ are outgoing for $r > r_{\text{cr},1}$, become standing on sphere $r = r_{\text{cr},1}$ and ingoing for $r < r_{\text{cr},1}$.

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