EXAMINATION OF QUASI-LOCAL MASS FOR ASYMPTOTICALLY KERR SPACETIMES*

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The Hamiltonian definition of quasi-local mass is given. It is based on the multipole decomposition on Round Sphere or Rigid Sphere. The test of our quasi-local mass is performed for the Kerr spacetime.

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1. Introduction

Using a variational formulation of general relativity including boundary terms (see [1]), we introduce a definition of a Hamiltonian quasi-local mass. We choose initial-boundary data which leads to a well-posed initial-boundary problem after linearization. The quasi-local mass assigned to two-dimensional topologically spherical surfaces in the asymptotic region corresponds to the mass parameter m up to $O\left(\frac{1}{R^5}\right)$. In particular, Rigid and Round Spheres are analyzed. Additionally, a natural choice of control data related to angular momentum is needed to get the quasi-local mass with higher order precision.

2. The Kijowski variational formula

We wish to define a Hamiltonian dynamical system on a set of Lorentzian metrics on a four-dimensional manifold $\mathcal{M}$, assuming the existence of a three-dimensional spacelike surface $\Sigma \subset \mathcal{M}$ on which the metric approaches a Kerr metric, as one recedes to infinity along an end of Σ.

In order to present our results, we introduce some notation analogical to [1]. $\mathcal{M}$ is a four-dimensional spacetime equipped with a Lorentzian metric $g_{\mu\nu}$. Consider a spacetime domain Ω with a smooth timelike boundary such

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that $\mathcal{V} := \Omega \cap \Sigma$ is compact. Let $x^\mu = (x^0, x^k)$ be coordinates on Ω such that x^0 is a time coordinate constant on Σ and x^k are local coordinates on Σ. By $x^a = (x^0, x^A)$ we mean coordinates on a world tube $\partial\Omega$ with coordinates x^A on $\partial\mathcal{V}$. We distinguish $x^r \in \{x^k\}$ which is constant on $\partial\mathcal{V}$. The decomposition into the time and space coordinates is performed. ν and ν^k are the lapse function and the shift vector, respectively. We have chosen a vector field $X = \partial_0$ as a vector field associated with the Hamiltonian flow. One of the main results of the paper [1] is the following formula:

$$\int\limits_{\mathcal{V}} \left[\partial_0 P^{kl}\delta g_{kl} - \partial_0 g_{kl}\delta P^{kl}\right] + 2\int\limits_{\partial\mathcal{V}} \left[\partial_0\lambda\delta\alpha - \partial_0\alpha\delta\lambda\right]$$
$$= -\int\limits_{\partial\mathcal{V}} \left[2\nu\delta\boldsymbol{Q} - 2\nu^A\delta\boldsymbol{Q}_A + \nu\boldsymbol{Q}^{AB}\delta g_{AB}\right], \tag{1}$$

where $P^{kl} := \sqrt{|\det g_{mn}|}\left(K\tilde{g}^{kl} - K^{kl}\right)$ — ADM momentum, $\lambda := \sqrt{\det g_{AB}}$ — the two-dimensional area element on $\partial\mathcal{V}$, $\alpha := \operatorname{arsinh}\left(\frac{g^{0r}}{\sqrt{|g^{00}g^{rr}|}}\right)$ — the hyperbolic angle between $\partial\Omega$ and $\partial\mathcal{V}$. $Q^{ab} := \sqrt{|\det g_{ef}|}(S\tilde{g}^{ab} - S^{ab})$ is the ADM counterpart, analog to the ADM momentum on the world tube, where $S_{ab} = -\frac{\Gamma^r_{ab}}{\sqrt{g^{rr}}}$ — the extrinsic curvature tensor of $\partial\Omega$ embedded in $\mathcal{M}$. One can define the following densities on $\partial\mathcal{V}$: $\boldsymbol{Q} := \nu Q^{00}$, $\boldsymbol{Q}_A := Q^0{}_A = Q^{0b}g_{bA}$, $\boldsymbol{Q}^{AB} := \frac{1}{\nu}Q_{CD}\tilde{\tilde{g}}^{CA}\tilde{\tilde{g}}^{DB}$. The main idea is to investigate a quasi-local mass on two-dimensional surfaces which mimic spheres in flat Minkowski spacetime. In this section, we propose two particular classes of such surfaces: Rigid Spheres and Round Spheres. The reach geometrical structure of the selected surfaces gives a potentially large chance to set specialized gauge conditions. Our concept is to make a multipole decomposition and set appropriate gauge conditions for each pole. For further details, see [3]. The surfaces are defined as follows:

Round Sphere is a topological two-dimensional sphere whose inner geometry is round. It means that the curvature scalar is constant

$$R^{(2)}(g|_{\mathcal{S}}) = \text{const} \tag{2}$$

or in other words, that it is a constant curvature space. In this case, there exists a coordinate system in which the induced, two-dimensional metric is, up to a multiplicative constant, a standard spherical metric σ_{AB}, hence, the multipole expansion is well-defined.

Rigid Sphere: Let Σ be a Riemannian three-manifold and let $\mathcal{S} \subset \Sigma$ be a two-dimensional, topological sphere. We say that $\mathcal{S}$ is a Rigid Sphere[1] if its mean extrinsic curvature k satisfies the following equation:

$$(I - P_{md})k = 0 \tag{3}$$

which means that the mean curvature vanishes modulo mono-dipole part. We use the multipole splitting. $()_m$, $()_d$ and $()_w$ denote respectively the monopole part, the dipole part and the higher multipoles part. Additionally, each covector on the unit sphere can be decomposed into its closed (gradient) $||$ part and coclosed (dual of the gradient) $\perp$ part. Formula (1) can be represented as

$$\begin{aligned}
&\int_{\partial\mathcal{V}} \left[2\nu\delta\boldsymbol{Q} - 2\nu^A\delta\boldsymbol{Q}_A + \nu\boldsymbol{Q}^{AB}\delta g_{AB}\right] = 16\pi\delta\boldsymbol{H}_{\partial\mathcal{V}} - \int_{\partial\mathcal{V}} \left\{k_m^2\delta\left[\lambda\left(\frac{\nu}{k}\right)_m\right]\right\} \\
&-2\int_{\partial\mathcal{V}} \left[\left({}_\perp\nu_A\right)_d \delta\left({}_\perp\boldsymbol{Q}_d^A - c\lambda\,{}_{||}\nu_d^A\right) + \left({}_\perp\nu_A\right)_w \delta\left({}_\perp\boldsymbol{Q}_w^A\right) + \left({}_{||}\nu_A\right)_w \delta\left({}_{||}\boldsymbol{Q}_w^A\right)\right] \\
&+\int_{\partial\mathcal{V}} \left[\left({}_{||}\nu_A\right)_d \delta\left({}_{||}\boldsymbol{Q}_d^A\right) + \left(\frac{\nu}{k}\right)_d \delta\left(k_d^2\right) + \left(\frac{\nu}{k}\right)_w \delta\left(k_w^2\right) + \boldsymbol{Q}^{AB}\delta g_{AB}\right] .
\end{aligned} \tag{4}$$

On the RHS of the above formula, all the terms except $\delta\boldsymbol{H}_{\partial\mathcal{V}}$ vanish due to boundary conditions on $\partial\mathcal{V}$ which correspond to gauge or real degrees of freedom respectively. See [3] for detailed analysis. The quasi-local Hamiltonian is the following:

$$\boldsymbol{H}_{\partial\mathcal{V}} = \frac{1}{16\pi}\int_{\partial\mathcal{V}} \left\{\lambda\left(\frac{\nu}{k}\right)_m \left[\left(k^2\right)_m - \left(k_0^2\right)_m\right] + \left({}_\perp\nu_A\right)_d\, {}_\perp\boldsymbol{Q}_d^A\right\} , \tag{5}$$

where k_0 is the mean curvature of the reference frame. It is a mean curvature of $\partial\mathcal{V}$ embedded in three-dimensional Euclidean space.

3. Tests for Kerr spacetime based on asymptotic series

We start with the Kerr spacetime in the Boyer–Lindquist coordinates. In both cases of Spheres, the surface can be described as $R = f(r,\theta) = \text{const}$ in terms of asymptotic series[2].

[1] Rigid Spheres are an improved idea of Huang proposed by Gittel, Jezierski, Kijowski and Łęski in [2]. Originally, the authors define the multipole expansion based on equilibrated coordinates.

[2] We do not know the method which solves the Round Sphere equation or the Rigid Sphere equation directly. The asymptotic series solution is proposed $R = \sum_{j=1}^{-\infty} c_j(\theta) r^j = c_1(\theta) r + c_0(\theta) + \frac{c_{-1}(\theta)}{r} + \ldots$ which is calculated symbolically up to satisfied precision (some $c_n(\theta)$).

3.1. Results for Round Spheres

It is convenient to perform a two-dimensional transformation of coordinates $(r, \theta) \to (R, \Theta)$, where R is constant on the Sphere and Θ is defined such that it fulfills the conditions $g_{\Theta\Theta} = R^2$ and $g_{\varphi\varphi} = R^2 \sin^2 \Theta$. We restrict ourselves to the first seven terms of the asymptotic series for R. The accuracy of the Round Sphere condition (2) is $R^{(2)} = \frac{2}{R^2} + O\left(\frac{1}{R^9}\right)$. Compute an asymptotic series of the Hamiltonian (5) for the Round Sphere of radius R

$$\boldsymbol{H}_{\partial\mathcal{V}} = m - \frac{242 m^2 a^4}{40 R^5} + O\left(\frac{1}{R^6}\right). \tag{6}$$

Full calculations are available in [3].

3.2. Results for Rigid Spheres

We restrict ourselves to the first seven terms of the asymptotic series for R. The accuracy of the Rigid Sphere condition (3) is $k^2 = \frac{4}{R^2} - \frac{8m}{R^3} + O\left(\frac{1}{R^{10}}\right)$. Rigid Spheres do not possess a natural multipole structure. We receive it by introducing the Equilibrated Coordinates (EC), a specially selected conformally spherical coordinates introduced in [2].

Let h_{AB} be an induced metric on $\mathcal{S}$. We perform a transformation of coordinates into (Θ, φ), such that the transformed metric on $\mathcal{S}$ is conformally spherical and the transformation preserves axial symmetry for the φ coordinate. It has the form of $h_{AB}\mathrm{d}x^A\mathrm{d}x^B = \Psi^2\left[\mathrm{d}\Theta^2 + \sin^2\Theta\mathrm{d}\varphi^2\right]$. Using EC, the multipole decomposition is well-defined. Compute an asymptotic series of the Hamiltonian (5) for the Rigid Sphere of radius R

$$\boldsymbol{H}_{\partial\mathcal{V}} = m + \frac{9 m^4 a^2}{R^5} + O\left(\frac{1}{R^6}\right). \tag{7}$$

Full calculations are available in [3].

4. Comparison of the obtained results and conclusions

We would like to point out some observations:

1. In both cases, the obtained Hamiltonians (6) and (7) have the properties:
   - The mass term m is exposed in comparison with the first unwanted deviation term for big values of radial coordinate R.
   - The standard deviation related to the angular momentum ma does not occur. It results from the three facts: specific choice of surface, restriction only to the lower-rank multipoles in the Hamiltonian and the correction which comes from the $\nu^A \boldsymbol{Q}_A$ term.

2. The coordinates R and Θ for Round and Rigid Spheres have different meaning although in both cases, the leading term is the same. Formulas (6) and (7) cannot be compared directly. The unwanted corrections to the mass in Eqs. (6) and (7) have the same rank but opposite signs. There is expected that it vanishes for the Mixed-Geometry Spheres, the new class of surfaces which mixes the properties of Round Spheres and Rigid Spheres. The work is in progress.

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