A NOTE ON EVGENY M. LIFSHITZ HISTORICAL CONTRIBUTION*

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(Received March 22, 2017)

An analytical solution of exact perturbation equations in the flat radiation-dominated relativistic cosmology posed by Evgeny M. Lifshitz in 1946 is found. From this, we obtain exact form for the scale-dependent growth factor function which is important in observational cosmology as a useful tool of model testing.

DOI:10.5506/APhysPolBSupp.10.419

The dynamics of inhomogeneous perturbations of the Friedmann–Lemaître–Robertson–Walker (FLRW) metric play a key role in the explanation of the large scale structure of the Universe and of Cosmic Microwave Background Radiation (CMB) anisotropies. The first perturbation scheme in cosmology was worked out as early as in 1946 by Lifshitz [1] (see also [2–5] for historical remarks and later development).

* Talk presented by A. Borowiec at the 3rd Conference of the Polish Society on Relativity, Kraków, Poland, September 25–29, 2016.

Following [3,4], where one finds more complete presentation, let us consider small perturbations around spacially flat FLRW Universe $\mathrm{d}t^2 - a^2(t)\mathrm{d}l^2$ in regions which are small in comparison to the scale factor a. In that case, the spatial line element of the spacetime metric can be written as

$$\mathrm{d}l^2 = a^2(\eta)\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right), \tag{1}$$

where x, y, z are the Cartesian coordinates. We take conformal time η as a time coordinate, *i.e.* $c\,\mathrm{d}t = a(\eta)\mathrm{d}\eta$ and t denotes the cosmic time. The matter source is described by the perfect fluid energy-momentum tensor $T_{\mu\nu} = (p+\epsilon)u_\mu u_\nu + p g_{\mu\nu}$, where the pressure $p = p(\epsilon)$ is assumed to be a function of the energy density ϵ by a suitable equation of state. Linearized equations for small perturbations take the form of

$$\delta R^\mu_\nu - \frac{1}{2}\delta^\mu_\nu \delta R = \frac{8\pi k}{c^4}\delta T^\mu_\nu\,, \tag{2}$$

where imposing a synchronous reference system, one finds

$$\delta T^i_j = -\delta^i_j \frac{\mathrm{d}p}{\mathrm{d}\epsilon}\delta T^0_0\,, \qquad \delta T^i_0 = \delta u^i\,, \qquad \delta T^0_0 = \delta\epsilon\,. \tag{3}$$

In particular, δu^i do not vanish and the reference system is not longer comoving. For adiabatic perturbations, $\sqrt{\frac{\mathrm{d}p}{\mathrm{d}\epsilon}} = c_\mathrm{s}$ is called the speed of sound.

We are interested in the relative change of energy density (overdensities) called also fractional density perturbations or density contrast

$$\delta_m := \frac{\delta\epsilon}{\epsilon} = \frac{c^4}{16\pi k\epsilon}\left(\delta R^0_0 - \frac{1}{2}\delta R\right) = \frac{c^4}{16\pi k\epsilon a^2}\left(h^{j,i}_{i,j} - h^{,i}_{,i} + \frac{2a'}{a}h'\right), \tag{4}$$

where h_{ij} denote perturbations of the spatial metric (1) and $' \equiv \frac{\mathrm{d}}{\mathrm{d}\eta}$. It turns out that these perturbations can be expressed as plane waves

$$h^i_j = \lambda(\eta)P^i_j + \mu(\eta)Q^i_j\,, \tag{5}$$

where $Q^i_j = \frac{1}{3}\delta^i_j e^{i\boldsymbol{n}\boldsymbol{r}}$ and $P^i_j = (\frac{1}{3}\delta^i_j - \frac{n^i n_j}{n^2})e^{i\boldsymbol{n}\boldsymbol{r}}$, while the vector $\boldsymbol{n}$ is a normalized wave vector $\boldsymbol{k} = \boldsymbol{n}/a$. In that terms, equations (2) are[1] $(n = |\boldsymbol{n}|)$

$$\begin{aligned} \lambda'' + \frac{2a'}{a}\lambda' - \frac{n^2}{3}(\lambda+\mu) &= 0\,, \\ \mu'' + \mu'\frac{a'}{a}\left(2 + 3\frac{\mathrm{d}p}{\mathrm{d}\epsilon}\right) + \frac{n^2}{3}(\lambda+\mu)\left(1 + 3\frac{\mathrm{d}p}{\mathrm{d}\epsilon}\right) &= 0\,. \end{aligned} \tag{6}$$

[1] Some formulas from [1] have been corrected in [3,4].

We want to explore the Universe at radiation dominated epoch: $p = \epsilon/3$, $a = a_1\eta$, so equations (6) are now

$$\begin{aligned} \lambda'' + \frac{2}{\eta}\lambda' - \frac{n^2}{3}(\lambda+\mu) &= 0\,, \\ \mu'' + \frac{3}{\eta}\mu' + \frac{2n^2}{3}(\lambda+\mu) &= 0\,. \end{aligned} \tag{7}$$

System (7) can be written as a third order linear ODE (from now on $v = \frac{n^2}{3}$ for short) for the function $\lambda'(\eta)$

$$\lambda^{(4)} + \frac{5}{\eta}\lambda^{(3)} + \left(\frac{2}{\eta^2} + v\right)\lambda'' + \left(\frac{v}{\eta} - \frac{2}{\eta^3}\right)\lambda' = 0 \tag{8}$$

which has a general solution in the form of

$$\lambda'(\eta) = \frac{d_1}{\eta} + \frac{d_3\cos\sqrt{v}\eta - d_2\sin\sqrt{v}\eta}{\eta^2}\,. \tag{9}$$

We are now in position to calculate density contrast (4), which has a form of plane wave with time-dependent amplitude $\mathfrak{a}_{\boldsymbol{n}}(\eta)$ for each Fourier mode

$$\delta_m(\boldsymbol{n},\eta,\boldsymbol{r}) = \frac{c^4}{24\pi k\epsilon a^2}\left(n^2(\lambda+\mu) + \frac{3a'}{a}\mu'\right)e^{i\boldsymbol{n}\boldsymbol{r}} = \mathfrak{a}_{\boldsymbol{n}}(\eta)e^{i\boldsymbol{n}\boldsymbol{r}}\,, \tag{10}$$

where

$$\mathfrak{a}_{\boldsymbol{n}}(\eta) = \frac{c^4}{8\pi k}\,\frac{\lambda'\left(n^2\eta^2-6\right) + \lambda''\left(n^2\eta^3+6\eta\right) + 3\eta^2\lambda^{(3)}}{n^2\eta} \tag{11}$$

or

$$\mathfrak{a}_{\boldsymbol{n}}(\eta) \sim \frac{\left(2d_2v\eta + d_3\sqrt{v}\left(2-v\eta^2\right)\right)\sin\sqrt{v}\eta - \sqrt{v}\left(2d_3\sqrt{v}\eta + d_2\left(v\eta^2-2\right)\right)\cos\sqrt{v}\eta - 2d_1/v}{\eta^2}\,. \tag{12}$$

The expansion around small $v\eta \ll 1$ gives

$$\mathfrak{a}_{\boldsymbol{n}}(\eta) \sim -\frac{2\left(d_1/v - d_2\sqrt{v}\,\right)}{\eta^2} - \frac{d_3v^2}{3}\eta + \frac{d_2v^{\frac{5}{2}}}{4}\eta^2 + O(\eta)^3 \tag{13}$$

which agrees with [3, 4] provided $d_1 = d_2\sqrt{v^3}$. In the limit of $v\eta \mapsto \infty$ (late times approximation), we are left with oscillating terms (*cf.* [1, 3])

$$\mathfrak{a}_{\boldsymbol{n}}(\eta) \sim d_2\sqrt{v}\cos\sqrt{v}\eta + d_3\sqrt{v}\sin\sqrt{v}\eta\,. \tag{14}$$

Let us notice that from now on we may study the evolution of the linear scalar perturbations (4). One introduces a very useful tool for investigations of the evolutionary scenarios of the Universe expansion, that is, the growth factor function and its parametrization

$$f := \frac{\mathrm{d}\ln\delta_m}{\mathrm{d}\ln a} = \Omega_m(a)^\gamma\,, \tag{15}$$

where γ is the growth index. It is often used with the scale-independent approximation for the perturbation modes. For the LCDM model, this function was a subject of many papers, *e.g.* [6–11]. From the observations, one obtains data for the growth factor f from the Lyman-alfa forests and galaxy redshift distortions taken from [12–16]. Since at the radiation epoch $a \sim \eta$, it would be useful to rewrite formula (15) to the form

$$f = \frac{\eta}{\delta_m}\frac{\mathrm{d}\delta_m}{\mathrm{d}\eta} \tag{16}$$

which, in our case, equals to

$$\frac{\left(d_3 v\eta\left(4-v\eta^2\right)+2d_2\sqrt{v}\left(v\eta^2-2\right)\right)\cos\sqrt{v}\eta+\left(d_2 v\eta\left(v\eta^2-4\right)+2d_3\sqrt{v}\left(v\eta^2-2\right)\right)\sin\sqrt{v}\eta+4d_1}{\left(d_3\sqrt{v}\left(v\eta^2-2\right)-2d_2 v\eta\right)\sin\sqrt{v}\eta+\left(d_2\sqrt{v}\left(2-v\eta^2\right)-2d_3 v\eta\right)\cos\sqrt{v}\eta-2d_1}\,. \tag{17}$$

It would be challenging to obtain a similar result for more advanced cosmological scenarios, *e.g.* the ones presented in [17].

A.B. and M.S. were supported by the Polish National Science Center (NCN) project 2013/09/B/ST2/03455. A.W. is supported by the Polish National Science Center (NCN) project 2014/15/B/ST2/00089.

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