INHOMOGENEOUS CHIRAL CONDENSATES IN THE QCD PHASE DIAGRAM: CRITICAL OR LIFSHITZ POINT?*

M. Buballa

Theoriezentrum, Institut für Kernphysik, TU Darmstadt, Germany

S. Carignano

INFN, Laboratori Nazionali del Gran Sasso, Assergi (AQ), Italy

(Received February 16, 2017)

We discuss how the phase diagram of strong interaction matter is modified if inhomogeneous chiral condensates are allowed to form. In particular, we investigate the appearance of a Lifshitz point and the fate of the critical point in presence of an inhomogeneous phase.

DOI:10.5506/APhysPolBSupp.10.805

1. Introduction

The study of the phase diagram of quantum chromodynamics (QCD) is the object of intense investigations both from theory and experiments. While thanks to *ab initio* lattice calculations and heavy-ion collisions the properties of matter at finite temperatures and vanishing densities are now understood quite well, there is no consensus yet on the QCD phase structure at finite chemical potentials. In this region, no *ab initio* simulations are yet feasible, and most of our understanding comes from calculations performed within effective models. Of particular interest is the nature of the phase transition related to chiral symmetry, which is spontaneously broken in vacuum through the formation of a quark–antiquark condensate. At low temperatures, as the density of the system increases, the standard expectation corroborated by most model predictions is that chiral symmetry is restored via a first-order transition. Since, on the other hand, lattice results agree on the fact that at vanishing densities, there is no phase transition

* Presented at the "Critical Point and Onset of Deconfinement" Conference, Wrocław, Poland, May 30–June 4, 2016.

but rather a smooth crossover, this first-order phase boundary would have to end at a critical point (CP), which would then act as a cornerstone of the phase diagram.

On the other hand, the models predicting a first-order phase transition usually assume that the order parameter is spatially homogeneous. If this assumption is relaxed, several calculations within different frameworks suggest that an inhomogeneous phase with spatially modulated chiral condensates appears in the phase diagram, completely covering the first-order phase boundary of the homogeneous analysis (see [1] for a recent review). As a result, the CP could be replaced by a "Lifshitz point" (LP) at which the inhomogeneous phase and the two homogeneous phases with broken and restored chiral symmetry meet.

This leads to the question whether these model results are general features, which may then indicate a similar behavior in QCD, or whether they strongly depend on details of the models. In this contribution, we will therefore summarize the main results about the positions of CP and LP obtained within different effective chiral models of QCD and some of their common extensions.

2. Inhomogeneous phases and the Lifshitz point

Inhomogeneous chiral condensates have been studied within effective models such as the Nambu–Jona-Lasinio (NJL) model or the Quark–Meson (QM) model (Gell-Mann–Lévy model or linear sigma model with quarks). Within the mean-field approximation, the determination of the thermodynamic potential per unit volume at finite temperature and chemical potential for these models involves a functional trace over the inverse quark propagator, which depends on the scalar and pseudoscalar condensates (S and P, respectively). From a technical point of view, this trace becomes extremely involved if the mean-fields are allowed to be space-dependent, as the resulting quark Hamiltonian is no longer diagonal in momentum space. The problem has, therefore, not yet been solved in full generality, and one typically resorts to simplified Ansätze for the spatial dependence of the order parameters.

An alternative approach which can provide useful information on the nature of the CP and LP without specifying the functional form of the spatial modulation is a Ginzburg–Landau (GL) expansion of the thermodynamic potential Ω. For this, the thermodynamic potential Ω is expanded in powers of a small order parameter (a so-called "mass function") $M(\boldsymbol{x}) = m - 2G(S(\boldsymbol{x}) + iP(\boldsymbol{x}))$ and its derivatives

$$\Omega(M) = \Omega(0) + \frac{1}{V}\int \mathrm{d}^3x \left\{ \frac{1}{2}\gamma_2 |M(\boldsymbol{x})|^2 + \frac{1}{4}\left(\gamma_{4,a}|M(\boldsymbol{x})|^4 + \gamma_{4,b}|\nabla M(\boldsymbol{x})|^2\right) + \ldots \right\}, \tag{1}$$

where odd terms vanish in the chiral limit because of chiral and rotational symmetry. Higher-order terms, which are indicated by the ellipsis, are important for the stability of the system and therefore assumed to be positive.

The GL coefficients are functions of temperature and chemical potential and thus determine the phase structure of the model. In particular, if $\gamma_{4,b} > 0$, gradients of the mass function are disfavored and the analysis is reduced to the well-known case for homogeneous phases. One finds that the CP, where the first-order phase transition turns into second order, is determined by the condition

$$\mathrm{CP}: \ \gamma_2(T,\mu) = \gamma_{4,a}(T,\mu) = 0\,. \tag{2}$$

For $\gamma_{4,b} < 0$, on the other hand, inhomogeneous solutions, *i.e.*, solutions with non-zero gradients can become favored. One thus finds that the LP, where the homogeneous chirally broken and restored phases meet with the inhomogeneous one is given by the condition

$$\mathrm{LP}: \ \gamma_2(T,\mu) = \gamma_{4,b}(T,\mu) = 0\,. \tag{3}$$

The GL coefficients for the standard NJL model with scalar and pseudoscalar interactions have been calculated in [2], generalizing similar analyses from the $1+1$ dimensional Gross–Neveu model [3] to $3+1$ dimensions. The remarkable result was that the two fourth-order coefficients are equal, that is $\gamma_{4,a} = \gamma_{4,b}$, and therefore the LP coincides with the CP. The first-order transition line becomes then entirely covered by an inhomogeneous phase.

At this point, one might wonder about the robustness of this result. As already mentioned, GL analyses are rather general, in the sense that the specific form of the modulation does not need to be specified. Moreover, the result $\gamma_{4,a} = \gamma_{4,b}$ is independent of the choice of model parameters. It could however depend on the approximation scheme (here: mean-field approximation) and on the chosen model.

To investigate the dependence on the latter, in [4] some common extensions of the NJL model have been considered. As known since a long time, the inclusion of vector interactions has a very large effect on the CP, shifting its position to lower temperatures and eventually leading to its disappearance once the vector coupling G_V becomes sufficiently large [5]. It turned

out, however, that the behavior of the LP is rather different. In fact, the effect of the vector interaction on the GL coefficients associated with this point simply amounts to an effective shift in the chemical potential

$$\gamma_2\left(T,\mu;G_{\rm V}\right) = \gamma_2\left(T,\tilde{\mu};G_{\rm V}=0\right)\,, \tag{4}$$

$$\gamma_{4,b}\left(T,\mu;G_{\rm V}\right) = \gamma_{4,b}\left(T,\tilde{\mu};G_{\rm V}=0\right) \tag{5}$$

with

$$\tilde{\mu} = \mu - 2G_{\rm V}n \tag{6}$$

and n being the quark number density of the system. As a result, the LP is only shifted to higher chemical potentials, staying at the same temperature. On the other hand, the effect of the vector interaction on the GL coefficient $\gamma_{4,a}$, which is relevant for the CP, cannot simply be expressed through a chemical-potential shift. In this case, an additional correction term arises, leading to the already mentioned temperature shift of the CP as a function of $G_{\rm V}$. CP and LP thus split, as shown in Fig. 1 (left). Note, however, that the presence of the inhomogeneous phase may invalidate the GL analysis for the CP. In fact, for any $G_{\rm V} > 0$ the would-be CP lies inside of the inhomogeneous phase and has thus disappeared from the phase diagram.

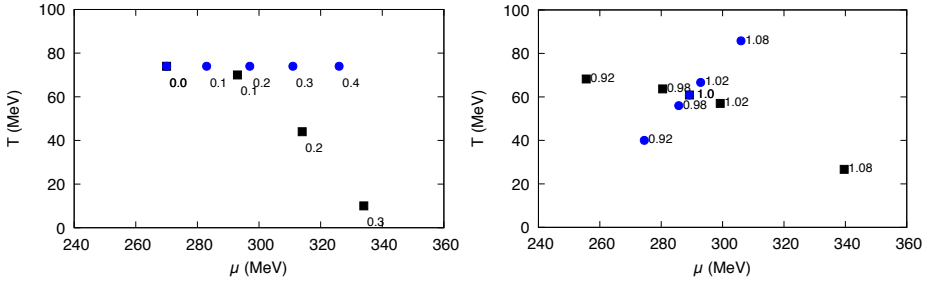


Fig. 1. (Color online) Position of the CP (black squares) and LP (blue dots) in the phase diagram. Left: effect of vector interactions in the NJL model (results are shown for different ratios $G_{\rm V}/G$). Right: effect of different m_σ in the QM model (results are shown for different ratios $m_\sigma/2M_q$).

The inclusion of an effective coupling with the Polyakov loop was also discussed in [4] within a PNJL model calculation. The main effect resulting from this model extension was found to be a stretch of the phase diagram in the T direction, leading to a shift of both, CP and LP, towards higher temperatures. A GL analysis suggests that the two points also split in this case, most likely in the opposite direction, so that the CP lies slightly outside of the inhomogeneous region if vector interactions are switched off. However, the exact determination of the coefficients turns out to be more cumbersome, and the net effect is numerically very small.

Finally, we discuss the relationship between the CP and LP in the QM model. This can also be performed within a GL analysis, as was done in [6] and [7]. The difference with the NJL model lies in an additional contribution to the GL coefficients given by the meson potential, which replaces the condensate term present in the NJL model and depends on the QM model parameters. The latter are fitted to reproduce vacuum phenomenology, typically represented by the pion decay constant f_π, the constituent quark mass M_q and the sigma meson mass m_σ. It was found that in the QM model, the positions of the CP and LP coincide only if the relation $m_\sigma = 2M_q$ (which is automatically fulfilled in the NJL model!) is enforced when fitting the parameters. More precisely, the splitting of the GL coefficients associated with the CP and LP is given by

$$\gamma_{4,a} - \gamma_{4,b} = 2\left(\frac{m_\sigma^2}{4M_q^2} - 1\right)\left(\frac{f_\pi^2}{M_q^2} - \frac{1}{2}\delta L_2\left(m_\sigma^2\right)\right), \tag{7}$$

where δL_2 is a finite loop integral, and the quantity in the first parenthesis vanishes when $m_\sigma = 2M_q$. The resulting splitting of the two points for different values of the ratio $m_\sigma/2M_q$ is shown in Fig. 1 (right).

3. Discussion

We have discussed the behavior of the CP and LP in the phase diagrams of various QCD-inspired models, allowing for an inhomogeneous phase where the chiral condensate is spatially non-uniform. While a GL analysis of the standard NJL model with scalar and pseudoscalar interactions reveals that the two points coincide, this result is not stable with respect to extensions or modifications of the model. In particular, their positions are extremely sensitive to the ratio $m_\sigma/2M_q$ in the QM model and to the value of the vector coupling constant. From this, one might conclude that the coincidence of CP and LP in the standard NJL model is merely an accident. However, in a Dyson–Schwinger QCD study, which was performed in Ref. [8], CP and LP also seem to agree, at least within numerical accuracy. A more careful investigation of this result in terms of a GL-like analysis would certainly be interesting.

M.B. acknowledges travel support by the COST action MP1304.

REFERENCES

[1] M. Buballa, S. Carignano, *Prog. Part. Nucl. Phys.* **81**, 39 (2015).
[2] D. Nickel, *Phys. Rev. Lett.* **103**, 072301 (2009).
[3] O. Schnetz, M. Thies, K. Urlichs, *Ann. Phys.* **314**, 425 (2004).
[4] S. Carignano, D. Nickel, M. Buballa, *Phys. Rev. D* **82**, 054009 (2010).
[5] K. Fukushima, *Phys. Rev. D* **78**, 114019 (2008).
[6] S. Carignano, M. Buballa, B.-J. Schaefer, *Phys. Rev. D* **90**, 014033 (2014).
[7] S. Carignano, M. Buballa, W. Elkamhawy, *Phys. Rev. D* **94**, 034023 (2016).
[8] D. Müller, M. Buballa, J. Wambach, *Phys. Lett. B* **727**, 240 (2013).