SCALE INVARIANT RESUMMED THERMAL PERTURBATIVE EXPANSIONS*

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We illustrate how our recently developed renormalization group optimized perturbation (RGOPT) efficiently resums perturbative expansions in thermal field theories. The residual renormalization scale dependence of optimized thermodynamical quantities is drastically improved as compared to either standard perturbative expansions, or related methods such as the screened perturbation or (resummed) hard-thermal-loop perturbation. Our approach is illustrated briefly for the nonlinear sigma model, as a toy model for thermal QCD. Finally, preliminary applications of RGOPT to hard thermal loop resummation for the QCD pressure are sketched.

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1. Introduction

For QCD at finite temperatures, lattice simulations have well-established the crossover 'transition' at about $T_c \simeq 170$ MeV [1]. However, the presently unsolved lattice sign problem, which occurs upon considering finite chemical potentials, prevents those methods from describing the more complete QCD phase diagram. A possible alternative is to use more analytical nonperturbative approximations, possibly addressing the problem of resumming the (notoriously badly convergent [2]) standard thermal perturbative expansion. One of these approximations is optimized perturbation (OPT) [3], which is briefly recalled in the next section. Similar approaches, like screened perturbation theory (SPT) [4] in the thermal context, have been extended to gauge theories, providing a resummation of hard thermal loop [5] perturbation theory (HTLpt) [6]. Unfortunately, even when restricted to only T-dependence, both SPT- and HTLpt-resummed thermodynamical quantities exhibit a residual renormalization scale dependence that unexpectedly *grows* at higher orders [7,8], even dramatically at three-loop order. Typically,

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the three-loop order QCD HTLpt pressure shows a good agreement [8] with lattice results down to low temperatures $T \simeq 2T_c$, for the 'central' scale value $\mu = 2\pi T$, but the agreement is lost as the renormalization scale is varied even by a moderate amount. An alternative reconciling OPT with renormalization group (RG) invariance has been proposed, dubbed renormalization group optimized perturbation theory (RGOPT). It was originally developed at vanishing temperatures [9, 10] and next, extended to finite temperature for the scalar ϕ^4 model, where is generically obtained the sought-after more moderate residual scale dependence [11]. In the next section, we recall the OPT (or similarly SPT) basic features, and our RG-compatible RGOPT version in Sec. 3. It is illustrated in Sec. 4 for the nonlinear sigma model in 1+1 dimensions, which shares many features with QCD, such as asymptotic freedom, the generation of a mass gap, and trace anomaly. Finally, some preliminary results for (pure gauge) QCD are sketched in Sec. 5. A summary and outlook are given in Sec. 6. More complete analyses will be given elsewhere for the NLSM [12] and QCD [13].

2. Optimized Perturbation Theory (OPT)

Consider the ϕ^4 model Lagrangian for concreteness

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{g}{4!}\phi^4\,. \tag{1}$$

The key feature of OPT is to reorganize the above Lagrangian (1) by "adding and subtracting" an *arbitrary* mass, treating one mass piece as an interaction. To perform this systematically, it is convenient to introduce an extra expansion parameter $0 \le \delta \le 1$, interpolating between $\mathcal{L}_{\rm free}$ and $\mathcal{L}_{\rm int}$, so that the mass is traded for an arbitrary variational parameter m [3]. Starting from any renormalized physical quantity $P(m, g)$, it is equivalent to reexpand it in powers of δ after substituting

$$m^2 \to m^2\,(1-\delta)^{2a}\,, \qquad g \to \delta\, g\,. \tag{2}$$

Note the exponent a in (2), reflecting a possibly more general interpolation prescription, but as we will see below, a is uniquely fixed from requiring [10,11] consistent renormalization group (RG) invariance properties. Applying (2) to a physical quantity $P(m, g)$, expanded in δ at some chosen order, and taking *afterwards* the limit $\delta \to 1$ to recover the *massless* theory, leaves a remnant m-dependence at any finite δ^k-order. The arbitrary mass parameter m is then fixed by an optimization (OPT) prescription

$$\frac{\partial}{\partial\, m} P^{(k)}(m, g, \delta = 1)|_{m\equiv\bar{m}} \equiv 0\,, \tag{3}$$

thus determining a nontrivial optimized mass $\bar{m}(g) \neq 0$.

3. Renormalization group consistency of OPT (RGOPT)

Before applying the δ-expansion (2), a first crucial step is to start from a (perturbatively) RG invariant physical quantity. However, this is generally not the case for the pressure: for example, the $\overline{\rm MS}$-scheme renormalized two-loop free energy (equivalently minus the pressure) of the scalar model is [7]

$$(4\pi)^2\mathcal{F}_0 = -\frac{m^4}{8}(3+2L) - \frac{T^4}{2}J_0(x) + \frac{g}{128\pi^2}\left[(L+1)m^2 - T^2 J_1(x)\right]^2 , \quad (4)$$

where $x = m/T$, $L \equiv \ln(\mu^2/m^2)$, μ the arbitrary renormalization scale, and

$$J_0(x) = \frac{16}{3}\int_0^\infty {\rm d}t \frac{t^4}{\sqrt{t^2+x^2}}\frac{1}{e^{\sqrt{t^2+x^2}}-1} , \qquad J_1(x) = -\frac{1}{2x}\partial J_0(x)/\partial x . \quad (5)$$

Now, applying the standard RG operator[1]

$$\mu\frac{\rm d}{{\rm d}\,\mu} = \mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + \gamma_m(g)\, m\frac{\partial}{\partial m} \quad (6)$$

to Eq. (4), leaves a remnant term already at lowest order, $-m^{4/2} + \mathcal{O}(g)$: a general feature which simply reflects that the vacuum energy of a massive theory has an anomalous dimension. Accordingly, perturbative RG invariance can be restored order by order from an appropriate finite vacuum energy subtraction term, $\mathcal{F}_0(m,g) \to \mathcal{F}_0(m,g) + \mathcal{E}_0(m,g)$

$$\mathcal{E}_0(m,g) = -m^4\left(\frac{s_0}{g} + s_1 + s_2 g + \cdots\right) . \quad (7)$$

For the scalar model, we find $s_0 = [2(b_0 - 4\gamma_0)]^{-1} = 8\pi^2$, $s_1 = -1$ *etc.* [11]. We are now ready to perform the OPT transformation from (2). Note that most previous OPT/SPT/HTLpt related approaches assumed $a = 1/2$ in (2), *i.e.* a linear δ-expansion. Also a well-known drawback is that Eq. (3) generally gives multiple mass gap solutions at increasing orders, and some being complex-valued. Thus, without insight on the nonperturbative behaviour, it can be difficult to select the right solution, and non-real solutions are embarrassing. Our construction [10, 11] differs in two respects. First, we combine OPT and RG properties, by requiring the (δ-modified) result to satisfy, in addition to Eq. (3), the perturbative RG equation

$$\mu\frac{\rm d}{{\rm d}\mu}\left(P^{(k)}(m,g,\delta=1)\right) = 0 . \quad (8)$$

[1] Our normalization here is $\beta(g) \equiv {\rm d}g/{\rm d}\ln\mu = b_0 g^2 + b_1 g^3 + \cdots$, $\gamma_m(g) = \gamma_0 g + \gamma_1 g^2 + \ldots$

Now, what appears overlooked in previous related approaches, is that after performing (2), perturbative RG invariance is generally badly lost, so that Eq. (8) gives a nontrivial additional constraint. In fact, RG invariance can only be restored for a unique value of the exponent a, fully determined by the universal (scheme-independent) first order RG coefficients

$$a = \gamma_0/b_0\,. \tag{9}$$

We stress that for a generic model at finite T, the above RGOPT construction has the following features: *(i)* an essentially unique solution of (3) at successive order, with a nontrivial mass gap $\bar{m}(g)$ already at one-loop order. *(ii)* The mass-gap and, therefore, the pressure are exactly scale-invariant by construction at one-loop order. *(iii)* A residual scale dependence unavoidably appears at higher orders, however much more moderate than in SPT and HTLpt related approaches. These features will be next illustrated with some new results for the nonlinear sigma model and QCD.

4. RGOPT of thermal nonlinear σ model (NLSM)

The O(N) NLSM Lagrangian in $1+1$ dimensions describes interactions of $N-1$ pions, with a mass perturbatively introduced to cure infrared divergences [14]. We are ultimately interested in the massless limit, where the Goldstone bosons acquire nonperturbatively a mass gap, such that the O(N) symmetry remains unbroken, in agreement with the Mermin–Wagner–Coleman theorem [15]. Applying the RGOPT up to two-loop order with NLSM b_0, γ_0 *etc.* values, Fig. 1 compares the one- and two-loop RGOPT NLSM pressure with large-N (LN) and two-loop SPT ones. The one-loop RGOPT pressure is exactly scale invariant, while the residual two-loop scale dependence improvement is about a factor 3 as compared to SPT.

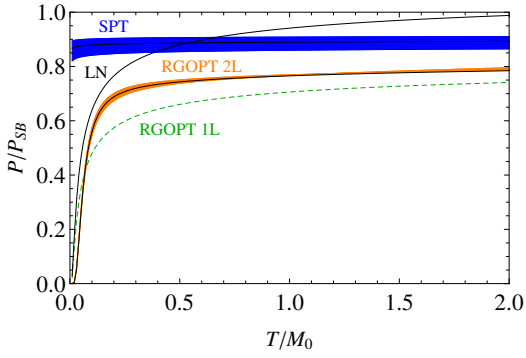


Fig. 1. NLSM pressure for different approximations, from [12], for a typical choice $N = 4$ and $g(\mu = M_0) = 1$ (shaded regions: scale-dependence $\pi T < \mu < 4\pi T$).

Moreover, at two-loop order, RGOPT compares well [12] with NLSM lattice results [16], and also gives a realistic trace anomaly, in contrast with SPT or HTLpt, reflecting in two dimensions the breaking of conformal invariance.

5. Thermal (pure glue) QCD: hard thermal loop

In this section, we only sketch some preliminary results for thermal QCD (pure gluon), see [13] for more details. The OPT procedure (2) operates on the gauge-invariant nonlocal[2] HTL effective Lagrangian [5]

$$\mathcal{L}_{\rm QCD}({\rm gauge}) - \frac{m^2}{2}\,{\rm Tr}\left[G_{\mu\alpha}\left\langle\frac{y^\alpha y^\beta}{(y\,D)^2}\right\rangle_y G^\mu_\beta\right], \tag{10}$$

which describes a thermal gluon mass $m^2 \sim \alpha_{\rm S}T^2$, plus many more hard thermal loop contributions modifying gluon propagator and vertices very nontrivially. The HTLpt pressure has been calculated up to 3-loop $\alpha_{\rm S}^2$ (NNLO) order [8], where reasonable agreement with lattice simulations down to $T \sim 2$–$3\,T_{\rm c}$ is found for some renormalization scale choice, but an issue of HTLpt is the drastic increase of scale dependence at NNLO order.

Our main RGOPT improvement is the crucial RG invariance maintained at all stages of calculations: first, before performing (2), it is perturbatively restored by a subtraction in the pressure

$$P_{\rm HTLpt} \to P_{\rm HTLpt} - m^4\left(\frac{s_0}{\alpha_{\rm S}} + s_1 + \cdots\right),$$

reflecting its anomalous dimension. Second, we use our RG-compatible modification of the nonperturbative interpolation, taking (2) with $a = \gamma_0/b_0$, where the gluon mass anomalous dimension γ_0 is evaluated from its (available) counterterm. As expected, the one-loop RGOPT pressure is exactly scale-invariant, similarly to the previous models. At 2-loop order, a moderate scale dependence reappears, similar to the NLSM case, but with a factor $\sim$ 2–3 improvement with respect to 2-loop order HTLpt (see Fig. 2). On general grounds, the RGOPT scale dependence should further improve at 3-loop order, therefore a drastic improvement is expected as compared to 3-loop HTLpt: because RGOPT at $\mathcal{O}(\alpha_{\rm S}^k)$ implies that $\bar m(\mu)$ appears first at $\mathcal{O}(\alpha_{\rm S}^{k+1})$ for *any* $\bar m$, and since $\bar m^2 \sim \alpha_{\rm S}T^2$ and $P_{\rm RGOPT} \simeq \bar m^4/\alpha_{\rm S} + \cdots$, the leading μ-dependence should appear at $\mathcal{O}(\alpha_{\rm S}^{k+2})$. However, to determine the low $T \sim T_{\rm c}$ genuine RGOPT pressure shape, one needs higher order terms of $\mathcal{O}(m^4\alpha_{\rm S})$, $\mathcal{O}(m^4\alpha_{\rm S}^2)$, which implies new calculations of 2- and 3-loop HTL integrals.

[2] In (10), $D^\mu = \partial^\mu - igA^\mu$, $y^\mu = (1, \hat{\boldsymbol{y}})$, $\hat y^2 = 1$, and $\langle\cdots\rangle$ means angular averaging.

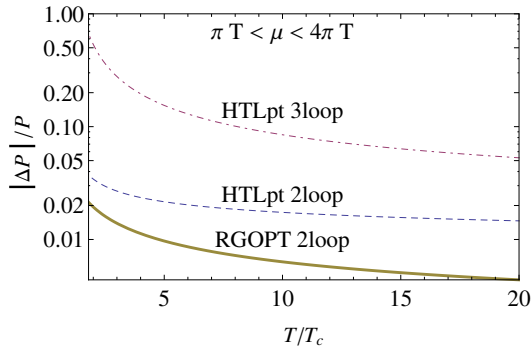


Fig. 2. Relative scale variation of pure glue pressure for RGOPT *versus* HTLpt.

6. Summary and outlook

Our RG-compatible version of OPT, RGOPT, gives an efficient alternative to related SPT/HTLpt thermal variational approaches, by maintaining or restoring RG invariance at all calculation stages. Therefore, the residual scale dependence remains moderate and is expected on general grounds to decrease at higher orders. Moreover, RGOPT appears to capture a more 'nonperturbative' behaviour already at two-loop order with, for instance, a realistic trace anomaly (*i.e.* with a maximum) obtained in the NLSM case [12]. Since our approach is generic, one expects to extend such variational methods to the full QCD thermodynamics, in particular for exploring the phase diagram also at finite densities.

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