

RELATIVISTIC HYDRODYNAMICS OF PARTICLES WITH SPIN $1/2^*$

WOJCIECH FLORKOWSKI^{a,b,c}, BENGT FRIMAN^d, AMARESH JAISWAL^{d,e}
ENRICO SPERANZA^{d,f}

^aH. Niewodniczański Institute of Nuclear Physics, Polish Academy of Sciences
Kraków, Poland

^bJan Kochanowski University, Kielce, Poland

^cExtreMe Matter Institute EMMI, GSI, Darmstadt, Germany

^dGSI Helmholtzzentrum für Schwerionenforschung, Darmstadt, Germany

^eSchool of Physical Sciences, National Institute of Science Education
and Research, HBNI, Jatni-752050, India

^fInstitut für Kernphysik, Technische Universität Darmstadt, Darmstadt, Germany

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A new hydrodynamic framework for particles with spin $1/2$, based solely on the conservation laws for charge, energy, momentum and angular momentum, is discussed.

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1. Introduction

In this paper, we report on recent work [1], where a novel hydrodynamic framework for particles with spin $1/2$ was introduced. The renewed interest in hydrodynamics of spinning particles is based on two facts: first, relativistic hydrodynamics forms the basic framework that is used to describe space-time evolution of matter created in relativistic heavy-ion collisions, studied experimentally at RHIC and the LHC [2], second, recently, measurements of particle polarization in heavy-ion collisions have become available [3]. Thus, it is tempting to combine these two topics to explore polarization effects in the context of hydrodynamic models (for a recent review of this and other related issues see, for example, Ref. [4] and references therein).

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2. Local equilibrium distribution functions

The main physics input for our approach is the definition of local equilibrium distribution functions for particles (plus signs) and antiparticles (minus signs) given in [5]

$$f_{rs}^+(x, p) = \frac{1}{2m} \bar{u}_r(p) X^+ u_s(p), \quad f_{rs}^-(x, p) = -\frac{1}{2m} \bar{v}_s(p) X^- v_r(p). \quad (1)$$

Here, $r, s = 1, 2$ are spin indices, u_r and v_s are bispinors, and X^\pm are the four-by-four matrices

$$X^\pm = \exp [\pm \xi(x) - \beta_\mu(x) p^\mu] M^\pm, \quad (2)$$

where

$$M^\pm = \exp \left[\pm \frac{1}{2} \omega_{\mu\nu}(x) \hat{\Sigma}^{\mu\nu} \right]. \quad (3)$$

Here, we use the notation $\beta^\mu = u^\mu/T$ and $\xi = \mu/T$, with the temperature T , chemical potential μ , and the fluid four-velocity u^μ ($u \cdot u = 1$). The quantity $\omega_{\mu\nu}$ is the polarization tensor, while $\hat{\Sigma}^{\mu\nu}$ is the spin operator expressed by the Dirac gamma matrices, $\hat{\Sigma}^{\mu\nu} = (i/4)[\gamma^\mu, \gamma^\nu]$.

It is convenient to express the polarization tensor $\omega_{\mu\nu}$ in terms of the four-vectors k^μ and ω^μ

$$\omega_{\mu\nu} \equiv k_\mu u_\nu - k_\nu u_\mu + \epsilon_{\mu\nu\beta\gamma} u^\beta \omega^\gamma. \quad (4)$$

We can assume that both k_μ and ω_μ are orthogonal to u^μ ($k \cdot u = \omega \cdot u = 0$), hence

$$k_\mu = \omega_{\mu\nu} u^\nu, \quad \omega_\mu = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \omega^{\nu\alpha} u^\beta. \quad (5)$$

We also define the dual polarization tensor

$$\tilde{\omega}_{\mu\nu} \equiv \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} \omega^{\alpha\beta} = \omega_\mu u_\nu - \omega_\nu u_\mu + \epsilon^{\mu\nu\alpha\beta} k_\alpha u_\beta. \quad (6)$$

It follows that $\frac{1}{2} \omega_{\mu\nu} \omega^{\mu\nu} = k \cdot k - \omega \cdot \omega$ and $\frac{1}{2} \tilde{\omega}_{\mu\nu} \omega^{\mu\nu} = 2k \cdot \omega$. Using the constraint

$$k \cdot \omega = 0, \quad (7)$$

we find the compact form

$$M^\pm = \cosh(\zeta) \pm \frac{\sinh(\zeta)}{2\zeta} \omega_{\mu\nu} \hat{\Sigma}^{\mu\nu}, \quad (8)$$

where

$$\zeta \equiv \frac{1}{2} \sqrt{k \cdot k - \omega \cdot \omega}. \quad (9)$$

3. Basic physical observables

The knowledge of the equilibrium distribution functions (1) allows us to compute the basic physical observables such as the charge and energy density, pressure, and entropy density. For the charge current, we use the definition of Refs. [5,6]

$$N^\mu = \int \frac{d^3p}{2(2\pi)^3 E_p} p^\mu [\text{tr}(X^+) - \text{tr}(X^-)] = nu^\mu, \tag{10}$$

where “tr” denotes the trace over spinor indices and n is the charge density

$$n = 4 \cosh(\zeta) \sinh(\xi) n_{(0)}(T) = 2 \cosh(\zeta) \left(e^\xi - e^{-\xi} \right) n_{(0)}(T). \tag{11}$$

Here, $n_{(0)}(T) = \langle (u \cdot p) \rangle_0$ is the number density of spinless, neutral Boltzmann particles, obtained using the thermal average

$$\langle \dots \rangle_0 \equiv \int \frac{d^3p}{(2\pi)^3 E_p} (\dots) e^{-\beta \cdot p}, \tag{12}$$

where $p^0 = E_p = \sqrt{m^2 + \mathbf{p}^2}$ is the particle energy.

In the next step, we calculate the energy-momentum tensor, again following Refs. [5,6]

$$T^{\mu\nu} = \int \frac{d^3p}{2(2\pi)^3 E_p} p^\mu p^\nu [\text{tr}(X^+) + \text{tr}(X^-)] = (\varepsilon + P)u^\mu u^\nu - P g^{\mu\nu}. \tag{13}$$

The energy density and pressure in (13) are given by the formulas

$$\varepsilon = 4 \cosh(\zeta) \cosh(\xi) \varepsilon_{(0)}(T) \tag{14}$$

and

$$P = 4 \cosh(\zeta) \cosh(\xi) P_{(0)}(T), \tag{15}$$

respectively. In analogy with the particle density $n_{(0)}(T)$, we define the auxiliary quantities $\varepsilon_{(0)}(T) = \langle (u \cdot p)^2 \rangle_0$ and $P_{(0)}(T) = -(1/3) \langle [p \cdot p - (u \cdot p)^2] \rangle_0$. We note that the energy-momentum tensor (13) is symmetric and has the structure characterizing perfect fluids.

For the entropy current, we use a straightforward generalization of the Boltzmann expression

$$S^\mu = - \int \frac{d^3p}{2(2\pi)^3 E_p} p^\mu \left(\text{tr} [X^+ (\ln X^+ - 1)] + \text{tr} [X^- (\ln X^- - 1)] \right). \tag{16}$$

This leads to the entropy density which satisfies the equation

$$s = u_\mu S^\mu = \frac{\varepsilon + P - \mu n - \Omega w}{T}, \quad (17)$$

where $\Omega = \zeta T$ and

$$w = 4 \sinh(\zeta) \cosh(\xi) n_{(0)}. \quad (18)$$

The last equation suggests that Ω can be used as a thermodynamic variable of the grand canonical potential, in addition to T and μ . Taking the pressure P to be a function of T, μ and Ω , $P = P(T, \mu, \Omega)$, one finds

$$s = \left. \frac{\partial P}{\partial T} \right|_{\mu, \Omega}, \quad n = \left. \frac{\partial P}{\partial \mu} \right|_{T, \Omega}, \quad w = \left. \frac{\partial P}{\partial \Omega} \right|_{T, \mu}. \quad (19)$$

4. Hydrodynamic equations

Hydrodynamic equations are first-order differential equations for the Lagrange multipliers appearing in the local equilibrium distribution functions. Since we use constraint (7) and introduce Ω to parametrize the contraction $\omega_{\mu\nu}\omega^{\mu\nu}$, ten independent functions of space and time are needed for a complete description. These are chosen as: $T(x)$, $\mu(x)$, $\Omega(x)$, three independent components of $u^\mu(x)$, and the four remaining independent components of $\omega^{\mu\nu}(x)$.

The conservation of energy and momentum implies that

$$\partial_\mu T^{\mu\nu} = 0. \quad (20)$$

This equation can be split into two parts, one longitudinal and the other transverse with respect to u^μ

$$\partial_\mu [(\varepsilon + P)u^\mu] = u^\mu \partial_\mu P \equiv \frac{dP}{d\tau}, \quad (21)$$

$$(\varepsilon + P) \frac{dw^\mu}{d\tau} = (g^{\mu\alpha} - u^\mu u^\alpha) \partial_\alpha P. \quad (22)$$

Evaluating the derivative on the left-hand side of the first equation, one finds

$$T \partial_\mu (su^\mu) + \mu \partial_\mu (nu^\mu) + \Omega \partial_\mu (wu^\mu) = 0. \quad (23)$$

The term in the middle of the left-hand side vanishes due to charge conservation

$$\partial_\mu (nu^\mu) = 0. \quad (24)$$

Thus, in order to have conservation of entropy in our system, $\partial_\mu (su^\mu) = 0$ (for the perfect-fluid description we are aiming at), we demand that

$$\partial_\mu (wu^\mu) = 0. \quad (25)$$

Equations (20), (24) and (25) form a closed system of six equations for six unknowns: $T(x)$, $\mu(x)$, $\Omega(x)$ and three components of $u^\mu(x)$. Since they do not determine the time evolution of the individual components of the polarization tensor, we dub them the equations for the hydrodynamic background.

5. Spin dynamics

Our approach is based on the conservation of the angular momentum in the form of $\partial_\lambda J^{\lambda,\mu\nu} = 0$, where $J^{\lambda,\mu\nu} = L^{\lambda,\mu\nu} + S^{\lambda,\mu\nu}$ with $L^{\lambda,\mu\nu} = x^\mu T^{\nu\lambda} - x^\nu T^{\mu\lambda}$ being the orbital angular momentum and $S^{\lambda,\mu\nu}$ the spin tensor. Since the energy-momentum tensor (13) is symmetric, the conservation law $\partial_\lambda J^{\lambda,\mu\nu} = 0$ implies conservation of the spin tensor $S^{\lambda,\mu\nu}$ [7]

$$\partial_\lambda S^{\lambda,\mu\nu} = 0. \quad (26)$$

For $S^{\lambda,\mu\nu}$, we use the form discussed in [8]

$$S^{\lambda,\mu\nu} = \int \frac{d^3p}{2(2\pi)^3 E_p} p^\lambda \text{tr} \left[(X^+ - X^-) \hat{\Sigma}^{\mu\nu} \right] = \frac{w u^\lambda}{4\zeta} \omega^{\mu\nu}. \quad (27)$$

Using the conservation law for the spin density and introducing the rescaled polarization tensor $\bar{\omega}^{\mu\nu} = \omega^{\mu\nu}/(2\zeta)$, we find

$$u^\lambda \partial_\lambda \bar{\omega}^{\mu\nu} = \frac{d\bar{\omega}^{\mu\nu}}{d\tau} = 0. \quad (28)$$

Since $\bar{\omega}^{\mu\nu}$ is antisymmetric, Eq. (28) with the normalization condition

$$\bar{\omega}_{\mu\nu} \bar{\omega}^{\mu\nu} = 2 \quad (29)$$

yields five independent equations. If condition (7) is fulfilled on the initial hypersurface, it remains fulfilled at later times, provided Eq. (28) holds. Hence, Eq. (28) used with (7) and (29) yields four additional equations that are needed to determine the space-time evolution of a spinning fluid. In Ref. [1], we have shown that this framework has a vortex-like solution that corresponds to global equilibrium studied in Refs. [5, 8].

6. Closing remarks

In this work, we have described a new hydrodynamic approach to relativistic perfect-fluid hydrodynamics of particles with spin 1/2. The system of hydrodynamic equations follows directly from the conservation laws for

charge, energy, momentum and angular momentum. An important ingredient of our approach is the form of the spin tensor defined by Eq. (27) that allows for the construction of a consistent system of equations. We note that form (27) differs from those used in [5] and [6], respectively.

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