MATTER–ANTIMATTER COEXISTENCE METHOD FOR FINITE DENSITY QCD*

Hideo Suganuma

Department of Physics, Graduate School of Science, Kyoto University
Kitashirakawaoiwake, Sakyo, Kyoto 606-8502, Japan

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We propose a "matter–antimatter coexistence method" for the finite-density lattice QCD, aiming at a possible solution of the sign problem. In this method, we consider matter and antimatter systems on two parallel $\mathbf{R}^4$-sheets in five-dimensional Euclidean space-time. For the matter system M with a chemical potential $\mu \in \mathbf{C}$ on a $\mathbf{R}^4$-sheet, we also prepare the antimatter system $\bar{M}$ with $-\mu^*$ on the other $\mathbf{R}^4$-sheet shifted in the fifth direction. In the lattice QCD formalism, we introduce a correlation term between the gauge variables $U_\nu \equiv e^{iagA_\nu}$ in M and $\tilde{U}_\nu \equiv e^{iag\tilde{A}_\nu}$ in $\bar{M}$, such as $S_\lambda \equiv \sum_{x,\nu} 2\lambda\{N_{\rm c} - {\rm Re\ tr}[U_\nu(x)\tilde{U}_\nu^\dagger(x)]\} \simeq \sum_x \frac{1}{2}\lambda a^2\{A_\nu^a(x) - \tilde{A}_\nu^a(x)\}^2$ with a real parameter λ. In the limit of $\lambda \to \infty$, a strong constraint $\tilde{U}_\nu(x) = U_\nu(x)$ is realized, and the total fermionic determinant is real and non-negative. In the limit of $\lambda \to 0$, this system goes to two separated ordinary QCD systems with the chemical potential of μ and $-\mu^*$. On a finite-volume lattice, if one takes a large enough value of λ, $\tilde{U}_\nu(x) \simeq U_\nu(x)$ is realized and there occurs a phase cancellation approximately between two fermionic determinants in M and $\bar{M}$, which is expected to suppress the sign problem and to make the lattice calculation possible. For the obtained gauge configurations of the coexistence system, matter-side quantities are evaluated through their measurement only for the matter part M. By the calculations with gradually decreasing λ and their extrapolation to $\lambda = 0$, physical quantities in finite density QCD are expected to be estimated.

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1. Introduction

The lattice QCD Monte Carlo calculation has revealed many aspects of the QCD vacuum and hadron properties in both zero and finite temperatures. At finite density, however, lattice QCD is not yet well-investigated, because of a serious problem called the "sign problem" [1, 2], which originates from the complex value including minus sign of the QCD action and

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the fermionic determinant at finite density, even in the Euclidean metric [3]. In fact, the Euclidean QCD action $S[A,\psi,\bar{\psi};\mu]$ at finite density with the chemical potential μ is generally complex

$$S\left[A,\psi,\bar{\psi};\mu\right] = S_G[A] + \int \mathrm{d}^4x \left\{\bar{\psi}(\not{D} + m + \mu\gamma_4)\psi\right\} \in \mathbf{C}\,, \tag{1}$$

with the gauge action $S_{\mathrm{G}}[A] \in \mathbf{R}$ and covariant derivative $D^\nu \equiv \partial^\nu + igA^\nu$. Then, the action factor cannot be identified as a probability density in the QCD generating functional, unlike ordinary lattice QCD calculations.

In this paper, aiming at a possible solution of the sign problem, we propose a new approach of a "matter–antimatter coexistence method" for lattice QCD at finite density with a general chemical potential $\mu \in \mathbf{C}$.

2. Matter–antimatter coexistence method

Our strategy is to use a cancelation of the phase factors of the fermionic determinants between a matter system with μ and an antimatter system with $-\mu^*$, and our method is based on the general property [3]

$$S\left[A,\psi,\bar{\psi};\mu\right]^* = S\left[A,\psi,\bar{\psi};-\mu^*\right] \tag{2}$$

for the Euclidean QCD action $S[A,\psi,\bar{\psi};\mu]$ in the presence of the chemical potential $\mu \in \mathbf{C}$. Actually, the fermionic kernel D_{F} corresponding to $\not{D} + m$ generally satisfies $D_{\mathrm{F}}^\dagger = \gamma_5 D_{\mathrm{F}} \gamma_5$ in lattice QCD, so that one finds

$$\left[\bar{\psi}(D_{\mathrm{F}} + \mu\gamma_4)\psi\right]^* = \bar{\psi}\left(D_{\mathrm{F}} - \mu^*\gamma_4\right)\psi\,, \tag{3}$$

which leads to relation (2), and

$$\mathrm{Det}\left(D_{\mathrm{F}} + \mu\gamma_4\right)^* = \mathrm{Det}\left(D_{\mathrm{F}} - \mu^*\gamma_4\right)\,. \tag{4}$$

2.1. Definition and setup

In the "matter–antimatter coexistence method", we consider matter and antimatter systems on two parallel $\mathbf{R}^4$-sheets in five-dimensional Euclidean space-time. For the matter system M with a chemical potential $\mu \in \mathbf{C}$ on a $\mathbf{R}^4$-sheet, we also prepare the antimatter system $\bar{M}$ with $-\mu^*$ on the other $\mathbf{R}^4$-sheet shifted in the fifth direction, as shown in Fig. 1.

We put an ordinary fermion field $\psi(x)$ with the mass m and the gauge variable $U_\nu(x) \equiv e^{iagA_\nu(x)}$ at $x \in \mathbf{R}^4$ on the matter system M, and we put the other fermion field $\Psi(x) \equiv \psi(x+\hat{5})$ with the same mass m and the gauge variable $\tilde{U}_\nu(x) \equiv e^{iag\tilde{A}_\nu(x)} \equiv U_\nu(x+\hat{5})$ on the antimatter system $\bar{M}$.

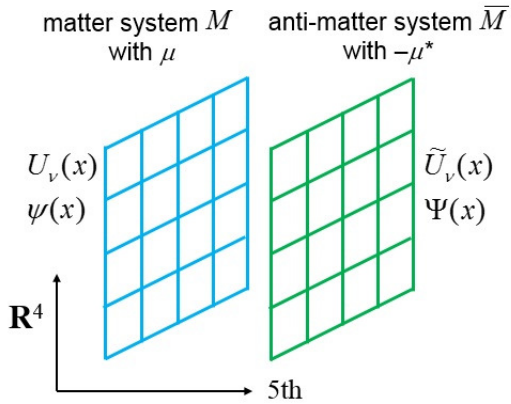

Fig. 1. The matter–antimatter coexistence system in five-dimensional Euclidean space-time. We put the matter system M with μ, $U_\nu(x)$ and $\psi(x)$ on a $\mathbf{R}^4$-sheet, and the antimatter system $\bar{M}$ with $-\mu^*$, $\tilde{U}_\nu(x) = U_\nu(x + \hat{5})$ and $\Psi(x) = \psi(x + \hat{5})$ on the other $\mathbf{R}^4$-sheet shifted in the fifth direction.

In the lattice QCD formalism, we introduce a correlation term between the gauge variables $U_\nu(x)$ in M and $\tilde{U}_\nu(x)$ in $\bar{M}$ at $x \in \mathbf{R}^4$, such as

$$S_\lambda \equiv \sum_{x,\nu} 2\lambda \left\{ N_{\rm c} - \mathrm{Re\ tr} \left[U_\nu(x) \tilde{U}_\nu^\dagger(x) \right] \right\} \tag{5}$$

with a real parameter λ (≥ 0), which connects two different situations: $\tilde{U}_\nu(x) = U_\nu(x)$ in $\lambda \to \infty$ and two separated QCD systems in $\lambda \to 0$. Near the continuum limit, this additional term becomes

$$S_\lambda \simeq \sum_x \tfrac{1}{2}\lambda a^2 \left\{ A_\nu^a(x) - \tilde{A}_\nu^a(x) \right\}^2 \simeq \int \mathrm{d}^4x \ \tfrac{1}{2}\lambda_{\rm phys} \left\{ A_\nu^a(x) - \tilde{A}_\nu^a(x) \right\}^2 \tag{6}$$

with $\lambda_{\rm phys} \equiv \lambda a^{-2}$.

In fact, the total lattice action in this method is written as

$$\begin{aligned} S \ = \ & S_{\rm G}[U] + \sum_x \bar{\psi} \left(D_{\rm F}[U] + \mu\gamma_4 \right) \psi + S_{\rm G}\left[\tilde{U}\right] + \sum_x \bar{\Psi} \left(D_{\rm F}\left[\tilde{U}\right] - \mu^*\gamma_4 \right) \Psi \\ & + \sum_{x,\nu} 2\lambda \left\{ N_{\rm c} - \mathrm{Re\ tr} \left[U_\nu(x) \tilde{U}_\nu^\dagger(x) \right] \right\} \end{aligned} \tag{7}$$

with the gauge action $S_{\rm G}[U] \in \mathbf{R}$ and the fermionic kernel $D_{\rm F}[U]$ in lattice QCD. After integrating out the fermion fields ψ and Ψ, the generating functional of this theory reads

$$
\begin{aligned}
Z &= \int DUe^{-S_{\rm G}[U]}{\rm Det}\left(D_{\rm F}[U]+\mu\gamma_4\right)\int D\tilde{U}e^{-S_{\rm G}[\tilde{U}]}{\rm Det}\left(D_{\rm F}\left[\tilde{U}\right]-\mu^*\gamma_4\right)\\
&\quad\times e^{-\sum_{x,\nu}2\lambda\left\{N_{\rm c}-{\rm Re\ tr}\left[U_\nu(x)\tilde{U}_\nu^\dagger(x)\right]\right\}}\\
&= \int DU\int D\tilde{U}e^{-\left(S_{\rm G}[U]+S_{\rm G}[\tilde{U}]\right)}{\rm Det}\left\{\left(D_{\rm F}[U]+\mu\gamma_4\right)\left(D_{\rm F}\left[\tilde{U}\right]-\mu^*\gamma_4\right)\right\}\\
&\quad\times e^{-\sum_{x,\nu}2\lambda\left\{N_{\rm c}-{\rm Re\ tr}\left[U_\nu(x)\tilde{U}_\nu^\dagger(x)\right]\right\}}. \qquad (8)
\end{aligned}
$$

In the continuum limit, this generating functional is expressed as

$$
\begin{aligned}
Z_{\rm cont} &= \int DA\int D\tilde{A}e^{-\left(S_{\rm G}[A]+S_{\rm G}[\tilde{A}]\right)}{\rm Det}\left\{\left(\not{D}+m+\mu\gamma_4\right)\left(\tilde{\not{D}}+m-\mu^*\gamma_4\right)\right\}\\
&\quad\times e^{-\int{\rm d}^4x\frac{1}{2}\lambda_{\rm phys}\left\{A_\nu^a(x)-\tilde{A}_\nu^a(x)\right\}^2} \qquad (9)
\end{aligned}
$$

with the continuum gauge action $S_{\rm G}[A]\in\mathbf{R}$ and $\tilde{D}^\nu\equiv\partial^\nu+ig\tilde{A}^\nu$.

In the practical lattice calculation with the Monte Carlo method, the fermionic determinant in Z is factorized into its amplitude and phase factor as

$$
\begin{aligned}
Z &= \int DU\int D\tilde{U}e^{-\left(S_{\rm G}[U]+S_{\rm G}[\tilde{U}]\right)}e^{-\sum_{x,\nu}2\lambda\left\{N_{\rm c}-{\rm Re\ tr}\left[U_\nu(x)\tilde{U}_\nu^\dagger(x)\right]\right\}}\\
&\quad\times\left|{\rm Det}\left\{\left(D_{\rm F}[U]+\mu\gamma_4\right)\left(D_{\rm F}\left[\tilde{U}\right]-\mu^*\gamma_4\right)\right\}\right|O_{\rm phase}\left[U,\tilde{U}\right], \qquad (10)
\end{aligned}
$$

and the phase factor of the total fermionic determinant

$$
O_{\rm phase}\left[U,\tilde{U}\right]\equiv e^{i{\rm arg}\left[{\rm Det}\left\{\left(D_{\rm F}[U]+\mu\gamma_4\right)\left(D_{\rm F}[\tilde{U}]-\mu^*\gamma_4\right)\right\}\right]}\in\mathbf{C} \qquad (11)
$$

is treated as an "operator" instead of a probability factor, while all other real non-negative factors in Z are treated as the probability density.

2.2. Property and procedure

The additional term S_λ connects the following two different situations as the two limits of the parameter λ.

1. In the limit of $\lambda\to\infty$, a strong constraint $\tilde{U}_\nu(x)=U_\nu(x)$ is realized, and the phase factors of two fermionic determinants ${\rm Det}(D_{\rm F}[U]+\mu\gamma_4)$ and ${\rm Det}(D_{\rm F}[\tilde{U}]-\mu^*\gamma_4)$ are completely cancelled, owing to Eq. (4). Therefore, the total fermionic determinant is real and non-negative

$$
{\rm Det}\left\{\left(D_{\rm F}[U]+\mu\gamma_4\right)\left(D_{\rm F}\left[\tilde{U}=U\right]-\mu^*\gamma_4\right)\right\}\geq 0\,, \qquad (12)
$$

and the sign problem is absent [4]. Note, however, that this system resembles QCD with an isospin chemical potential [5].

2. In the limit of $\lambda \to 0$, this system goes to two separated ordinary QCD systems with the chemical potential of μ and $-\mu^*$, although the cancellation of the phase factors cannot be expected between the two fermionic determinants $\mathrm{Det}(D_{\mathrm{F}}[U]+\mu\gamma_4)$ and $\mathrm{Det}(D_{\mathrm{F}}[\tilde{U}]-\mu^*\gamma_4)$ for significantly different $U_\nu(x)$ and $\tilde{U}_\nu(x)$, which are independently generated in the Monte Carlo simulation.

On a four-dimensional finite-volume lattice, if an enough large value of λ is taken, $\tilde{U}_\nu(x) \simeq U_\nu(x)$ is realized and there occurs the phase cancellation approximately between the two fermionic determinants $\mathrm{Det}(D_{\mathrm{F}}[U]+\mu\gamma_4)$ and $\mathrm{Det}(D_{\mathrm{F}}[\tilde{U}]-\mu^*\gamma_4)$ in M and $\bar{M}$, so that one expects a modest behavior of the phase factor $O_{\mathrm{phase}}[U,\tilde{U}]$ in Eq. (11), which leads to feasibility of the numerical lattice calculation with suppression of the sign problem.

Once the lattice gauge configurations of the coexistence system are obtained with the most importance sampling in the Monte Carlo simulation, matter-side quantities can be evaluated through their measurement only for the matter part M with μ.

By performing the lattice calculations with gradually decreasing λ and their extrapolation to $\lambda = 0$, we expect to estimate the physical quantities in finite density QCD with the chemical potential μ.

3. Summary, discussion and outlook

We have proposed a "matter–antimatter coexistence method" for the lattice calculation of finite density QCD. In this method, we have prepared matter M with μ and antimatter $\bar{M}$ with $-\mu^*$ on two parallel $\mathbf{R}^4$-sheets in five-dimensional Euclidean space-time, and have introduced a correlation term $S_\lambda \equiv \sum_{x,\nu} 2\lambda\{N_{\mathrm{c}} - \mathrm{Re}\ \mathrm{tr}[U_\nu(x)\tilde{U}_\nu^\dagger(x)]\} \simeq \sum_x \frac{1}{2}\lambda a^2\{A_\nu^a(x) - \tilde{A}_\nu^a(x)\}^2$ between the gauge variables $U_\nu = e^{iagA_\nu}$ in M and $\tilde{U}_\nu = e^{iag\tilde{A}_\nu}$ in $\bar{M}$. In the limit of $\lambda \to \infty$, owing to $\tilde{U}_\nu(x) = U_\nu(x)$, the total fermionic determinant is real and non-negative, and the sign problem is absent. In the limit of $\lambda \to 0$, this system goes to two separated ordinary QCD systems with the chemical potential of μ and $-\mu^*$.

For an enough large value of λ, $\tilde{U}_\nu(x) \simeq U_\nu(x)$ is realized and a phase cancellation approximately occurs between two fermionic determinants in M and $\bar{M}$, which is expected to suppress the sign problem and to make the lattice calculation possible. For the obtained gauge configurations of the coexistence system, matter-side quantities can be evaluated by their measurement only for the matter part M. By gradually reducing λ and the extrapolation to $\lambda = 0$, it is expected to obtain estimation of the physical quantities in finite density QCD with μ.

In this paper, we have demonstrated this method with taking S_λ in Eq. (5) as the simplest correlation between $U_\nu(x)$ in M and $\tilde{U}_\nu(x)$ in $\bar{M}$. In this method, however, there is some variety on the choice of the correlation between $U_\nu(x)$ and $\tilde{U}_\nu(x)$. For instance, it may be interesting to consider the other correlation term like

$$\begin{aligned}\bar{S}_\xi &\equiv \sum_x 8\xi \left(\sum_\nu \left\{ N_{\rm c} - {\rm Re\ tr} \left[U_\nu(x) \tilde{U}_\nu^\dagger(x) \right] \right\} \right)^3 \\ &\simeq \int {\rm d}^4x\ \tfrac{1}{8} a^2 \xi \left[\left\{ A_\nu^a(x) - \tilde{A}_\nu^a(x) \right\}^2 \right]^3 \end{aligned} \tag{13}$$

with a dimensionless non-negative real parameter ξ. At the classical level, this correlation is an irrelevant interaction and it gives vanishing contributions in the continuum limit $a \to 0$, like the Wilson term $-\frac{1}{2} ar\bar{\psi} D^2 \psi$.

The next step is to perform the actual lattice QCD calculation at finite density using this method. It would be useful to combine this method with the other known ways such as the hopping parameter expansion [6], the complex Langevin method [1] and the reweighting technique [2]. For example, if one utilizes the hopping parameter expansion, the quenched-level analysis becomes possible in this method, since the additional term S_λ only includes gauge variables.

Efficiency of this method would strongly depend on the system parameters, such as the space-time volume V, the quark mass m, the temperature T and the chemical potential μ. In any case, this method is expected to enlarge calculable area of the QCD phase diagram on (T, μ, m, V).

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