

POINT SYMMETRIES IN THE NUCLEAR SU(3) PARTNER GROUPS MODEL*

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The algebraic approach which allows for simulation of symmetries of a nucleus with respect to the laboratory and intrinsic frames is presented. The formalism is based on the partner groups (a group and the corresponding intrinsic group) idea. An illustrative example is related to the successful SU(3) Elliot nuclear model. An example of schematic Hamiltonian is chosen to have tetrahedral or octahedral symmetry.

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1. Introduction

The classical rotation is a well-understood phenomenon in which the orientation of a body is changing with time. On the other hand, the quantum rotation allows to determine only the probability of a given orientation of the “rotating” body and, in fact, the time dependence is not needed. In both cases, the rotational motion can be described as a motion on the rotation group manifold (the space of SO(3) group parameters). Usually, in the case of rotations, this manifold is parametrized by the set of Euler angles $\Omega = (\Omega_1, \Omega_2, \Omega_3)$. The quantum state space for this motion is not the group manifold itself but the space of square integrable complex functions of Euler angles denoted by $L^2(\text{SO}(3), d\mu(\Omega))$.

A link between abstract elements of the rotation group and the physical space of a nucleus is given by their action on the nuclear state space (they rotate nuclear states). On the other hand, one can consider two kinds of independent actions of the rotation group in the state space $L^2(\text{SO}(3), d\mu(\Omega))$ — the left action and the right action. A practical description of these

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actions was introduced in the textbook [2]. The left action is related to the laboratory frame and the corresponding properties of the physical system. The left action operators are “rotations” $R(\Omega) \in \text{SO}(3)$ of a nucleus with respect to the laboratory frame. The right action is related to the intrinsic properties of the physical system under consideration. The right action operators furnish the intrinsic rotation group denoted by $\overline{\text{SO}(3)}$. We call the group $\overline{\text{SO}(3)}$ the partner group of the group $\text{SO}(3)$ and *vice versa*.

The general definition of the intrinsic (partner) group \overline{G} for the group G can be found in [2]. For the completeness of our text, we sketch this idea below.

For each element g of the group G , one can define a corresponding operator \bar{g} in the linear group space \mathcal{L}_G as

$$\bar{g}|S\rangle = |Sg\rangle, \quad \text{for all } |S\rangle \in \mathcal{L}_G, \quad (1)$$

where the linear group space \mathcal{L}_G is defined as the linear space spanned by all possible formal linear combinations of the group elements

$$\mathcal{L}_G = \{|S\rangle : |S\rangle = \sum_{g \in G} c_g g, \quad \text{where } c_g \in \mathbb{C}\}. \quad (2)$$

The group formed by the collection of the operators \bar{g} is called the intrinsic group \overline{G} related to the group G . The pair G and \overline{G} are called partner groups. One of the most important properties of the intrinsic group \overline{G} is that this group commutes with its partner G

$$[G, \overline{G}] = 0. \quad (3)$$

The groups G and \overline{G} are anti-isomorphic. This property implies that the partner groups G and \overline{G} have similar algebraic structure and as a consequence representations, decompositions of the Kronecker products, Clebsch–Gordan coefficients *etc.*

2. Partner groups formalism

To explain the idea of the partner groups formalism, we follow the concept of the generalized rotor. The generalized quantum rotor is an extension of the standard second order rotor. It was used extensively with a great success in molecular physics [3–7] and later on developed, but not fully exploited, in nuclear physics [8–17].

The second order rotor Hamiltonian $H_{\text{rot}}^{(2)}$ is always expressed in terms of generators of the intrinsic group $\overline{\text{SO}(3)}$, the partner group to $\text{SO}(3)$. These generators are interpreted as angular momentum operators with respect to the intrinsic (rotating) frame [1]

$$H_{\text{rot}}^{(2)}(J_x, J_y, J_z) = \frac{\bar{J}_x^2}{2\mathcal{J}_x} + \frac{\bar{J}_y^2}{2\mathcal{J}_y} + \frac{\bar{J}_z^2}{2\mathcal{J}_z}. \quad (4)$$

The Cartesian form of this rotational Hamiltonian can be rewritten in a tensor form which is more convenient for its generalization

$$H_{\text{rot}}^{(2)}(\bar{J}_{-1}, \bar{J}_0, \bar{J}_{+1}) = h_0^0 \bar{T}_0^0 + h_0^2 \bar{T}_0^2 + h_2^2 (\bar{T}_2^2 + \bar{T}_{-2}^2), \quad (5)$$

where the coupling constants are functions of the inertia parameters \mathcal{J}_x , \mathcal{J}_y , \mathcal{J}_z

$$\begin{aligned} h_0^0 &= \frac{1}{2\sqrt{3}} \left(\frac{1}{\mathcal{J}_x} + \frac{1}{\mathcal{J}_y} + \frac{1}{\mathcal{J}_z} \right), & h_0^2 &= \frac{1}{\sqrt{6}} \left(\frac{1}{\mathcal{J}_z} - \frac{1}{2\mathcal{J}_x} + \frac{1}{2\mathcal{J}_y} \right), \\ h_2^2 &= \frac{1}{4} \left(\frac{1}{\mathcal{J}_x} - \frac{1}{\mathcal{J}_y} \right) \end{aligned} \quad (6)$$

and the appropriate tensors are defined as $\bar{T}_\mu^\lambda = (\bar{J} \otimes \bar{J})_\mu^\lambda$.

The generalized rotor model allows to simulate rotation-like motions with higher symmetries than those allowed for the standard second order rotor Hamiltonian. In this model, the Hamiltonian is a more complicated function of the laboratory and intrinsic components of the angular momentum operators (J_x, J_y, J_z) and $(\bar{J}_x, \bar{J}_y, \bar{J}_z)$, respectively,

$$H_{\text{rot}} = H_{\text{rot}}(J_x, J_y, J_z, \bar{J}_x, \bar{J}_y, \bar{J}_z) = \sum_{\lambda=0}^{\infty} \sum_{\mu=-\lambda}^{+\lambda} \left(h_\mu^\lambda (J^2) T_\mu^\lambda + h_\mu^{\lambda'} (J^2) \bar{T}_\mu^\lambda \right). \quad (7)$$

From the hermiticity condition of the Hamiltonian, one gets $h_{-\mu}^\lambda = (-1)^\mu h_\mu^{\lambda'}$ (the same for $h_\mu^{\lambda'}$ coefficient). The spherical laboratory (intrinsic) tensor operators T_μ^λ (\bar{T}_μ^λ) are built from the components of the laboratory (intrinsic) angular momentum operators

$$T_\mu^\lambda = \left(((J \otimes J)^2 \otimes J)^3 \dots \otimes J \right)_\mu^\lambda, \quad \bar{T}_\mu^\lambda = \left(((\bar{J} \otimes \bar{J})^2 \otimes \bar{J})^3 \dots \otimes \bar{J} \right)_\mu^\lambda. \quad (8)$$

One needs to remember that the laboratory and intrinsic angular momentum operators form the sets of generators of the partner rotation groups $SO(3)$ and $\overline{SO(3)}$, respectively. The configuration space for rotational motion is identified with the rotation group manifold $SO(3)$ usually parameterized by three Euler angles $\Omega = (\Omega_1, \Omega_2, \Omega_3)$. The Hilbert space of quantum states for the generalized rotor is the space of square integrable functions on the rotation group (the Euler angles are arguments of these functions),

$\mathcal{K}_{\text{SO}(3)} = \text{L}^2(\text{SO}(3), \text{d}\mu(\Omega))$, with the scalar product generated by the $\text{SO}(3)$ invariant Haar measure

$$\begin{aligned} \langle \psi_2 | \psi_1 \rangle &= \int_{\text{SO}(3)} \text{d}\mu(\Omega) \psi_2(\Omega)^* \psi_1(\Omega) \\ &= \frac{1}{8\pi^2} \int_0^{2\pi} \text{d}\Omega_1 \int_0^\pi \text{d}\Omega_2 \sin \Omega_2 \int_0^{2\pi} \text{d}\Omega_3 \psi_2(\Omega)^* \psi_1(\Omega). \end{aligned} \quad (9)$$

The commutation relation (3) among the elements of the partner groups implies that the corresponding generators also commute $[J_k, \bar{J}_l] = 0$. It means that the partner generators J_k and \bar{J}_l act on the state space \mathcal{K} independently. The Peter–Weyl theorem [18] introduces the orthonormal basis in the states space \mathcal{K} as a set of Wigner functions $r_{MK}^J(\Omega) = \sqrt{2J+1} D_{MK}^J(\Omega)^*$.

3. $\text{SU}(3) \times \overline{\text{SU}(3)}$ model with point symmetries

Using overgroup of rotational group of motion $G \supset \text{SO}(3)$ allows for the introduction of additional degrees of freedom. The $G = \text{SU}(3)$ nuclear model, which introduces the quadrupole–quadrupole separable interaction, is a simple but successful model [19, 20, 22–24]. Within the model of partner groups, it can be considered as an extension of the generalized rotor model. In addition to the rotational, it introduces the quadrupole degrees of freedom.

The partner groups $\text{SU}(3)$ and $\overline{\text{SU}(3)}$ allow for the analysis of additional symmetries, point symmetries for example, within the nuclear $\text{SU}(3)$ model.

Let us assume that the nuclear $\text{SU}(3)$ group is generated by three spherical (cyclic) components of the angular momentum operator $L_\mu(\theta)$, $\mu = -1, 0, +1$, and five components of the quadrupole tensor operator $Q_\nu(\theta)$, $\nu = -2, -1, 0, +1, +2$, where $\theta = (\theta_1, \theta_2, \dots, \theta_8)$ parametrizes elements of this group. In the following, for simplicity of notation, we do not write the dependence of generators on θ .

The transformation of generators from the group $\text{SU}(3)$ to the intrinsic group $\overline{\text{SU}(3)}$ (the transformation among the generators between the partner groups) can be obtain in a compact form as

$$\bar{X}_\mu = \sum_{\mu'=1}^8 \Delta_{\mu'\mu}^{(\Gamma_0)}(\theta) X_{\mu'}, \quad (10)$$

where $L_\mu = X_{\mu+2}$, $\mu = -1, 0, 1$, $Q_\nu = X_{\nu+6}$, $\nu = -2, \dots, 2$ and $\Delta^{(\Gamma_0)}(\theta)$ are matrices of the appropriate eight-dimensional irreducible representation equivalent to the adjoint representation of the nuclear $\text{SU}(3)$ group.

Here, the generators $\{\bar{L}_\mu\}$ are mixtures of the angular momentum L and the quadrupole Q operators, and cannot be interpreted as the angular momentum operator of the nucleus. The same remark concerns the intrinsic quadrupole operator $\{\bar{Q}_\mu\}$. The intrinsic generators fulfill the commutation relations for the group $\overline{SU(3)}$, which are very similar to those obtained by Elliot [23]

$$[\bar{L}_\mu, \bar{L}_\nu] = +\sqrt{2}(1\mu 1\nu | 1\mu + \nu) \bar{L}_{\mu+\nu}, \quad (11)$$

$$[\bar{Q}_\mu, \bar{L}_\nu] = +\sqrt{6}(1\mu 2\nu | 2\mu + \nu) \bar{Q}_{\mu+\nu}, \quad (12)$$

$$[\bar{Q}_\mu, \bar{Q}_\nu] = -3\sqrt{10}(2\mu 2\nu | 1\mu + \nu) L_{\mu+\nu}, \quad (13)$$

$$[L_\mu, \bar{L}_\nu] = [Q_\mu, \bar{Q}_\nu] = 0, \quad (14)$$

$$[L_\mu, \bar{Q}_\nu] = [Q_\mu, \bar{L}_\nu] = 0, \quad (15)$$

however, they differ by signs. The opposite sign in the commutation relations for the generators of the intrinsic groups is their characteristic feature.

In the past decade, the problem of existence of higher point symmetries in nuclei has been extensively investigated [25]. Usually, the point symmetries were related to shapes of nuclei and indirectly to their effective Hamiltonian. The geometrical interpretation is also supported for the generalized rotor. In this case, the transformation to the intrinsic group can be considered as the transformation to the rotating (intrinsic) frame. They are rigid-body transformations, *i.e.*, they do not change the shape of the body.

This is not the case in the $SU(3) \times \overline{SU(3)}$ partner groups model. However, this model allows for the investigation of the influence of the point symmetries within the nuclear $SU(3)$ approximation.

As an example, we consider the tetrahedral symmetry. The intrinsic Hamiltonian with a tetrahedral symmetry, different than the Hamiltonian of the generalized rotor, can be obtained as a sum of Hamiltonians constructed from the generators of $SU(3)$, generators of $\overline{SU(3)}$, and potentially the term describing coupling among them

$$H(Q, L, \bar{Q}, \bar{L}) = H'(L, Q) + H''(\bar{L}, \bar{Q}) + V(L, Q, \bar{L}, \bar{Q}). \quad (16)$$

Because the Hamiltonian H has to be invariant with respect to the rotation group $SO(3)$, the sub-Hamiltonian $H'(L, Q)$ has also to be rotationally invariant, *i.e.*, $SO(3) \subseteq \text{Sym}(H'(L, Q))$, where $\text{Sym}(H)$ denotes the symmetry group of the operator H . Using this notation, we can summarize the required symmetry properties: $SO(3) \subseteq \text{Sym}(H)$, $SO(3) \subseteq \text{Sym}(H')$, $SO(3) \subseteq \text{Sym}(V)$ and $\text{Sym}(H'') \subseteq \overline{SU(3)}$. The simplest $V = 0$ model Hamiltonian (16), having the required rotational symmetry, consists of the quadrupole–quadrupole interaction, the rotational term and other terms constructed

from generators of the partner intrinsic group. The sub-Hamiltonian H' used in the nuclear SU(3) model can be written as

$$\begin{aligned} H' &= \gamma C_2(\text{SU}(3)) - \kappa Q \cdot Q + \beta L \cdot L \\ &= (\gamma - \kappa) C_2(\text{SU}(3)) + (3\kappa + \beta) L^2, \end{aligned} \quad (17)$$

where the second order Casimir operator

$$C_2(\text{SU}(3)) = Q \cdot Q + 3L \cdot L. \quad (18)$$

The eigenvalues of the Casimir operator are $C_2(\lambda, \mu) = 4(\lambda^2 + \mu^2 + \lambda\mu + 3\lambda + 3\mu)$, where (λ, μ) labels the irreducible representation of SU(3) [20].

This implies the eigenvalues of H' to be $E'(\lambda, \mu) = (\gamma - \kappa)C_2(\lambda, \mu) + (3\kappa + \beta)L(L + 1)$.

The second part consists of tetrahedral or octahedral invariants. Examples of such invariants are listed below

$$H''_{3Q} = h_{3Q} \left((\bar{Q} \otimes \bar{Q})_2^3 - (\bar{Q} \otimes \bar{Q})_{-2}^3 \right), \quad (19)$$

$$H''_{3LQ} = h_{3LQ} \left((\bar{L} \otimes \bar{Q})_2^3 - (\bar{L} \otimes \bar{Q})_{-2}^3 \right), \quad (20)$$

$$H''_{4Q} = h_{4Q} \left(\sqrt{\frac{14}{5}} (\bar{Q} \otimes \bar{Q})_0^4 + (\bar{Q} \otimes \bar{Q})_{-4}^4 + (\bar{Q} \otimes \bar{Q})_4^4 \right). \quad (21)$$

As an example, we choose the tetrahedral (octahedral) H''_{4Q} “quadrupole–quadrupole”-type interaction. We call it the tetrahedral (octahedral) symmetry because $H = H' + H''_{4Q}$ is invariant with respect to spatial inversion and both isomorphic groups \bar{T}_d and \bar{O} cannot be distinguished. The tetrahedral and octahedral groups commute with the Hamiltonian, *i.e.*, they are its symmetry groups. To distinguish between both symmetry possibilities, one needs to know the transformation properties of states with respect to the space inversion (parity quantum numbers). This is dependent on the interpretation of parameters of the nuclear SU(3). We postpone the discussion of this problem to the upcoming paper.

Because the Clebsh–Gordan coefficients of the rotation group for the maximal and minimal projections are equal to 1, the term breaking SU(3) symmetry can be rewritten as

$$H''_{4Q} = h_{4Q} \left(\sqrt{\frac{14}{5}} (\bar{Q} \otimes \bar{Q})_0^4 + (\bar{Q}_2)^2 + (\bar{Q}_{-2})^2 \right). \quad (22)$$

The coupling constant h_{4Q} is here a real number which follows from the hermiticity requirement for Hamiltonians.

For calculations, we adopt the construction of the basis proposed in [26] and [27]. For the case of our partner groups, the corresponding basis is labelled with 8 quantum numbers

$$r_{f_i LM; \bar{f}_i \bar{L} \bar{M}}^{(\lambda, \mu)}(\theta) = \sqrt{\dim(\lambda, \mu)} \Delta_{f_i LM; \bar{f}_i \bar{L} \bar{M}}^{(\lambda, \mu)*}(\theta), \quad (23)$$

where $\Delta_{f_i LM; \bar{f}_i \bar{L} \bar{M}}^{(\lambda, \mu)*}(\theta)$ are matrix elements (Wigner functions) of irreducible representation (λ, μ) of the group $SU(3)$. One needs to note that the quantum numbers (λ, μ) label the irreducible representations of partner groups; these labels are related to the main invariants common for both groups. The first set of quantum numbers L, M describes the angular momentum and its projection in the laboratory frame. The corresponding pseudo-angular momenta \bar{L}, \bar{M} have the same properties as true angular momenta but they are observables independent of L, M . The quantum numbers f_i and \bar{f}_i distinguish equivalent irreducible representations of the groups $SO(3)$ and $\overline{SO(3)}$ in the chains $SU(3) \supset SO(3)$ and $\overline{SU(3)} \supset \overline{SO(3)}$, respectively.

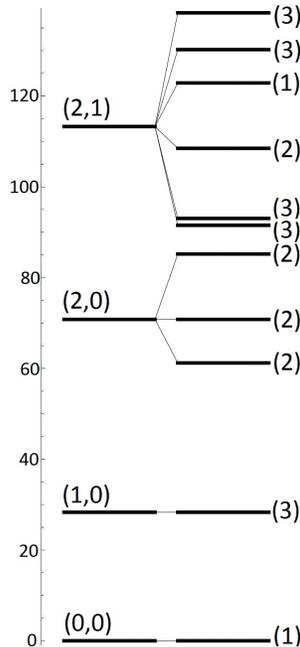


Fig. 1. Example of spectrum of Hamiltonian (24) for $\gamma' = 1.5$. The pair (λ, μ) labels the irreducible representations of the group $SU(3)$ and the label (n) , where $n = 1, 2, 3$ denote degeneration of eigenvalues due to the intrinsic tetrahedral/octahedral symmetry.

The most interesting part of the spectrum of the Hamiltonian H is now the intrinsic interaction term. To show the effect of this term on the energy spectrum, the eigenvalues H for $L = 0$,

$$H/h_{4Q} = \gamma' C_2(\text{SU}(3)) + H''_{4Q}/h_{4Q}, \quad (24)$$

are presented in Fig. 1. For $L > 0$, an additional rotational structure $L(L+1)$ is superimposed. This spectrum is calculated for the irreducible representations $(\lambda, \mu) = \{(0, 0), (1, 0), (2, 0), (2, 1)\}$. This Hamiltonian has the symmetry represented by the group $\text{SU}(3) \times \overline{T}_d$. The energy levels belonging to the representations $(\lambda, \mu) \in \{(0, 0), (1, 0)\}$ of the group $\text{SU}(3)$ are not affected by this type of tetrahedral (or octahedral) interaction. However, the higher energy levels, belonging to the representations $(\lambda, \mu) \in \{(2, 0), (2, 1)\}$, are split by the H''_{4Q} interaction. In Fig. 1, the label (n) denotes the dimensions of the irreducible representations of either \overline{T}_d or \overline{O} . As usually, these numbers are equal to the degenerations of the corresponding energy levels.

4. Conclusions

The formalism of partner groups allows for simulation of the intrinsic properties of quantum systems (also nuclei), including their intrinsic symmetries. In the example shown in this paper, the nuclear $\text{SU}(3)$ model is extended and allows for additional intrinsic structure, especially it allows to construct terms having required point symmetries. Molecular and nuclear physics are typical fields in which this type of models can be applied. It is a well-defined algebraic model which without very complicated and time-consuming calculations is able to reproduce energy spectra and transition amplitudes of some molecules and nuclei with given symmetries.

The highest symmetry of the Hamiltonian constructed within the partner groups model cannot be larger than the product of both groups. Such a Hamiltonian consists of two Casimir operators of both partner groups. In addition, these Hamiltonians should be invariant with respect to the generally required symmetries. These symmetries are dependent on a physical problem. In nuclear physics usually the rotational symmetry is required.

The partner groups approach is an interesting project allowing for the construction of algebraic nuclear models in which full power of the group theory methods can be used.

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