HIGGS PRODUCTION, DECAY, AND RELATED MATHEMATICAL OBJECTS*

Hjalte Frellesvig

Institute for Theoretical Particle Physics (TTP)
Karlsruhe Institute of Technology
Engesserstraße 7, 76128 Karlsruhe, Germany
and
Institute of Nuclear and Particle Physics, NCSR "Demokritos"
Patriarchou Gregoriou E. & Neapoleos 27, Agia Paraskevi, 15310, Greece

(Received January 22, 2018)

In these proceedings and the talk on which it is based, the author reviews the work he has participated in, during the last three years while being an ESR under the HiggsTools network. It is based on four peer-reviewed papers as well as some unpublished work. The paper is framed around the calculation of the planar Feynman integrals contributing to the two-loop correction to $H+j$ production in hadron colliders. That project (which had the calculation of the two-loop contribution to the decay width for $H \to Z\gamma$ as a spin-off) resulted in analytical expressions, of which some were expressible in terms of the function class of generalized polylogarithms, for which methods of reduction and evaluation will be discussed. Yet some of the the master integrals were not expressible in terms of that function class, for those elliptic integrals were needed, and a method useful to identify such cases, that of d-dimensional unitarity cuts, is discussed as well.

DOI:10.5506/APhysPolBSupp.11.365

1. Introduction

With the 2012 discovery of the Higgs boson and no discoveries of additional new elementary particles, the high-energy physics community is entering an era of precision physics. In particular, that will involve precision measurements of all properties of the Higgs boson, as the properties and interactions of that particle are most likely to form a window to new physics. Precise measurements necessitate precise calculations, if the size of the uncertainties of the experimental results are to remain higher than the theoretical uncertainties, and such precise calculations will involve the calculations of Feynman integrals with two or more loops. It should, however,

* Presented at the Final HiggsTools Meeting, Durham, UK, September 11–15, 2017.

be mentioned that the necessity for precise theoretical predictions is not the only reason to perform such calculations, they are also of interest from a more mathematical point of view, in the sense that they may help reveal mathematical structures underlying such loop-level amplitudes, and thereby all of nature.

2. Planar integrals for $H + j$ production

In Ref. [1], we presented the planar Feynman integrals needed for the calculation of the two-loop QCD correction to $H + j$ production. The calculation assumed light quarks to be massless, but retained the full dependence of the mass of the massive quark coupling to the Higgs boson, unlike other similar calculations [2–4].

The planar integrals were arranged into four different integral families (shown in Fig. 1), which means that each planar Feynman integral can be written as a linear combination of Feynman integrals for which the propagators form a subset of the seven propagators defining a family (up to permutations of the momenta of the external particles).

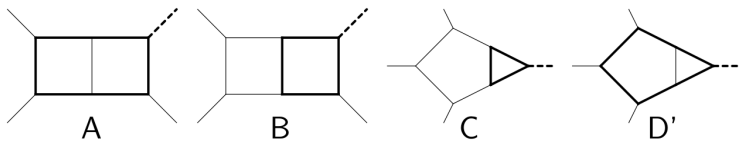

Fig. 1. The four planar integral families needed for $H + j$ production. Thin lines denote massless particles, thick lines particles with mass m_t, and the thick dashed line a particle with mass m_H.

The four integral families were evaluated using the method of differential equations [5–8] simplified by the use of canonical bases [9, 10] whenever possible. In total, the four families contain 125 Feynman integrals, which may be classified into three categories:

I. Integrals for which a canonical form was available, which allowed for an expression of the integral in terms of generalized polylogarithms or correspondingly log, Li_n, and $\mathrm{Li}_{2,2}$.

II. Integrals for which a canonical form was available, but where no expression in terms of polylogarithms could be found. Instead, an expression as an integral over logarithms and Li_2 was used.

III. Integrals for which no canonical form is available due to the result containing elliptic integrals.

For further discussion of these categories, see the following sections.

3. Generalized polylogarithms

The function class of generalized polylogarithms (GPLs), that show up in the first of the categories mentioned above, is defined recursively as [11,12]

$$G(a_1,\ldots,a_n;x) = \int_0^x \frac{\mathrm{d}z}{z-a_1} G(a_2,\ldots,a_n;z) \tag{1}$$

with

$$G(\underbrace{0,\ldots,0}_{n};x) = \frac{\log^n(x)}{n!} \qquad \text{and} \qquad G(;x) = 1\,. \tag{2}$$

This function class is a generalization of a number of functions known to show up in results for Feynman integrals, such as logarithms, the classical polylogarithms Li_n, and the harmonic polylogarithm $H_{\bar{m}}$ [13]. Generalized polylogarithms are subject to a large number of relations between each other [12, 14–20], and these relations allow for the reduction of any generalized polylogarithm to a member of some minimal set. For generalized polylogarithms with weight ≤ 4 (the weight of a GPL is the number of iterations, or correspondingly the number of a_i indices), that set may be chosen as the functions log, Li_2, Li_3, Li_4, and $\mathrm{Li}_{2,2}$, as shown in Ref. [16] together with explicit expressions for the reductions. Weight two has the first non-trivial reduction, which is given as

$$\begin{aligned} G(a,b,x) \;=\; & \log(1-\chi/\alpha)\log(\chi) + \mathrm{Li}_2(\chi/\alpha) - \mathrm{Li}_2(1/\alpha) \\ & + 2\pi i \log(\alpha)\mathrm{sgn}(\alpha) T(1,\chi,\alpha) \end{aligned} \tag{3}$$

with $\alpha = 1 - a/b$, $\chi = 1 - x/b$, and with $T(1,\chi,\alpha)$ being a function which evaluates to 1 whenever α is inside the triangle spanned (in the complex plane) by the points 0, 1, and χ, and to zero otherwise. At weights three and four, the corresponding relations are longer but of a similar nature, and they are all added to Ref. [16] in a directly usable `Mathematica` format.

Of the polylogarithmic functions left after the reduction, Li_n is well-studied and optimized methods for its evaluation are available. That is less so for $\mathrm{Li}_{2,2}$ which is defined as

$$\mathrm{Li}_{2,2}(x,y) = \sum_{i>j>0}^{\infty} \frac{x^i\, y^j}{i^2\, j^2}\,, \tag{4}$$

an expression which converges in the region where $|x| \leq 1$ and $|xy| \leq 1$. An optimized algorithm for the evaluation of $\mathrm{Li}_{2,2}$ is described in Ref. [16]. The algorithm works by mapping each $\mathrm{Li}_{2,2}$ for which the arguments are outside

the convergent region, to it, and additionally by finding ways to make the convergence faster close to $|xy| \approx 1$, where Eq. (4) converges slowly.

Implementations of the reduction and evaluation, made in respectively `Mathematica` and `C++`, are both made available with Ref. [16].

4. Higgs decay to $Z + \gamma$

No meaningful prediction for $H + j$ production can be made using only the planar integrals. Yet it turns out that the Feynman integrals needed for the calculation of the two-loop QCD correction to Higgs-decay into $Z + \gamma$ form a sub-set of the planar integrals discussed in the previous section. All these integral fall in category I, which means that they are expressible in terms of log, Li_n, and $\mathrm{Li}_{2,2}$ as discussed above. The result for the decay width for that process, including the two-loop QCD correction, were presented in Ref. [21] (see also Refs. [22, 23]), and it can be seen plotted in Fig. 2 as a function of the Higgs mass.

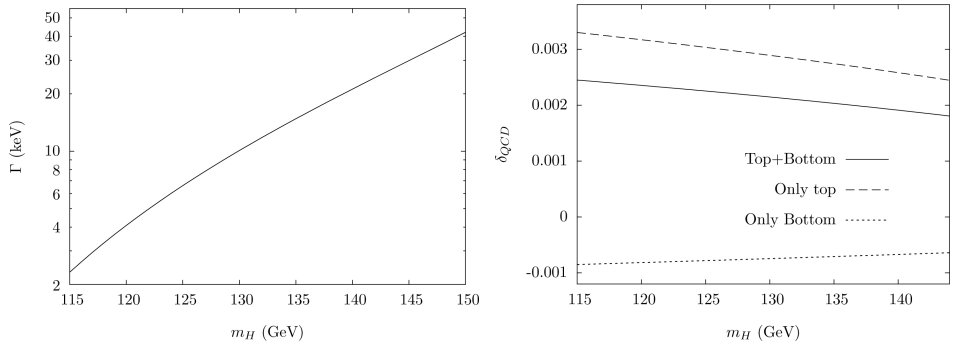

Fig. 2. Plots of the decay width Γ for $H \to Z\gamma$. δ_{QCD} in the right figure denotes the fraction of $\Gamma_{H\to Z\gamma}$ coming from the two-loop QCD correction. The figures are taken from Ref. [21].

5. *d*-dimensional unitarity cuts

Getting a set of Feynman integral into the canonical form used for the calculations in Ref. [1], and for most other recent calculations of Feynman integrals using the method of differential equations, is an art and a challenge. Various mathematical algorithms have been proposed [24–26], but the most successful method uses a consideration of the pole structure as revealed by generalized unitarity cuts [10].

A unitarity cut may be defined as a change of integration contour to a small circle (or in the multi-variable case a hypertorus) in the complex plane around the pole formed by a propagator in the Feynman integral. Unitarity

cuts have had much use in the analysis of one-loop diagrams, playing a large role in the so-called NLO revolution [27–31], but also at two-loop [32–35] and in other theories [36–38] there have been developments.

Most of the considerations cited above perform the unitarity cut in four dimensions. Yet in order to extract as much information as possible from the unitarity cut, for use in the extraction of a canonical form, it is desirable to perform the cut d-dimensionally.

One way of doing that, which is discussed in Ref. [39], consists of making a variable change from the momentum integration to an integration that uses the propagators of the Feynman integral as integration variables, in which case they are known as the Baikov variables [40–43]. In the Baikov variables, a Feynman integral has the (schematic) form

$$I \propto \int \frac{B(x)^q}{x_1^{a_1} \cdots x_n^{a_n}} \mathrm{d}^N x \,, \tag{5}$$

where the x_i are the Baikov variables, and where $B(x)$, which comes from Jacobian factors from the variable change, is known as the Baikov polynomial.

At one loop, $n = N = E + 1$ where E is the number of independent external momenta. At higher (L) loop orders, it is simple to get an expression for which $N = LE+L(L+1)/2$, but picking a clever order of the integrations can reduce that number in most cases. At higher loops, there is also no longer agreement between the number of integration variables N and the number of propagators n, and in most cases N will be a little larger than n.

In the Baikov variables, the unitarity cut procedure becomes almost trivial, as it can be written as

$$\int \frac{\mathrm{d}x}{x^a} \to \oint \frac{\mathrm{d}x}{x^a} \,, \tag{6}$$

where the latter integral can be trivially performed using the (multiple) residue theorem. Performing this procedure on all integration variables corresponding to propagators will give a constant for all one-loop Feynman integrals, and at higher loops, it will leave an $N - n$ dimensional integral to be done — a number that for two-loop cases is usually 1, occasionally 0, and never larger than 2.

Occasionally, the procedure of cutting all available propagators leaves an integral that evaluates to an elliptic integral — a class of functions that may be exemplified by the "complete elliptic integral of the first kind"

$$K(k) = \int_0^1 \frac{\mathrm{d}t}{\sqrt{(1-t^2)\,(1-k^2t^2)}} \,. \tag{7}$$

When this happens, one can deduce that the Feynman integral in question may not be expressed without such functions, which means that any attempt at finding a canonical form for that integral, or to express it in terms of GPLs, will be futile. Of the 125 Feynman integrals calculated in Ref. [1], only the eight that are seen in Fig. 3 had elliptic integrals in their result. They come in two different kinds. For the first four, the box-triangle topologies, the unitarity cut procedure discussed above reveals their elliptic structure — not so for the last four, the double-boxes. Their "ellipticity" appears because they couple to the box-triangles in the system of differential equations, so no new elliptic structures are introduced by the double-box topologies.

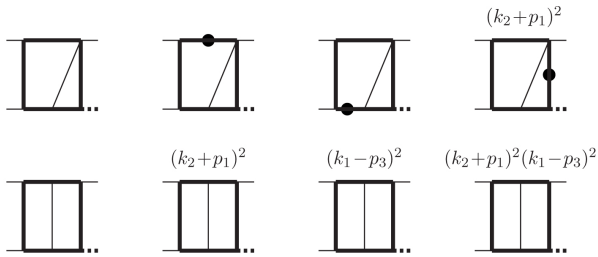

Fig. 3. The eight Feynman integrals from Ref. [1] that have elliptic integrals in their result. The figure is taken from Ref. [1].

Elliptic integrals in the context of Feynman integrals have a long history [33,44–47], most of it focusing on the so-called fully massive sunrise topology which is the simplest Feynman integral for which such a structure appears. See also Refs. [48–51] for newer works.

Several additional exiting new developments involving d-dimensional unitarity cuts and Baikov variables have been made in the recent years, see Refs. [52–58].

6. Discussion

In order to be able to calculate the NLO QCD correction to $H + j$ production, also the non-planar integrals are needed. This will introduce three additional families (E, F, and G), along with new elliptic structures. A result for these should become available in the not too distant future. One additional ingredient that will be needed in order to produce a result for the final cross section is the analytical continuation of the Feynman integrals to the physical region(s), something which may be a harder task than it sounds, particularly for the integrals in categories II and III. In addition, an electro-weak contribution to $H \to Z\gamma$ may appear.

The unitarity cut procedure discussed above is a way to distinguish the integrals that can be brought to canonical form (categories I and II), from those that cannot (category III). But is there an inherent property that

distinguishes category I and II, which it is possible to reveal using unitarity cuts or other methods? When an integral is in canonical form, it is known that the result can be formally expressed as Chen iterated integrals [59], but perhaps is it possible to specify a more minimal and specific set of functions that can be used to express all such cases, similarly to the set of log, Li_n, and $\mathrm{Li}_{2,2}$ for the case of generalized polylogarithms up to weight 4.

These considerations are both examples of natural continuations of the directions discussed on the previous pages.

I wish to thank the HiggsTools Initial Training Network PITN-GA-2012-316704, for the funding needed to produce the research outlined above. In addition, I wish to thank Roberto Bonciani, Vittorio Del Duca, Johannes Henn, Francesco Moriello, Costas Papadopoulos, Vladimir Smirnov, Damiano Tommasini, and Christopher Wever for collaboration on the mentioned projects.

REFERENCES

[1] R. Bonciani *et al.*, *J. High Energy Phys.* **1612**, 096 (2016).
[2] T. Neumann, C. Williams, *Phys. Rev. D* **95**, 014004 (2017) [arXiv:1609.00367 [hep-ph]].
[3] K. Melnikov, L. Tancredi, C. Wever, *J. High Energy Phys.* **1611**, 104 (2016) [arXiv:1610.03747 [hep-ph]].
[4] J.M. Lindert, K. Melnikov, L. Tancredi, C. Wever, *Phys. Rev. Lett.* **118**, 252002 (2017).
[5] A. Kotikov, *Phys. Lett. B* **254**, 158 (1991).
[6] E. Remiddi, *Nuovo Cim. A* **110**, 1435 (1997).
[7] T. Gehrmann, E. Remiddi, *Nucl. Phys. B* **580**, 485 (2000).
[8] C.G. Papadopoulos, *J. High Energy Phys.* **1407**, 088 (2014).
[9] J.M. Henn, *Phys. Rev. Lett.* **110**, 251601 (2013).
[10] J.M. Henn, *J. Phys. A* **48**, 153001 (2015).
[11] A.B. Goncharov, *Math. Res. Lett.* **5**, 497 (1998).
[12] A.B. Goncharov, M. Spradlin, C. Vergu, A. Volovich, *Phys. Rev. Lett.* **105**, 151605 (2010).
[13] E. Remiddi, J. Vermaseren, *Int. J. Mod. Phys. A* **15**, 725 (2000).
[14] C. Duhr, H. Gangl, J.R. Rhodes, *J. High Energy Phys.* **1210**, 075 (2012).
[15] J. Vollinga, S. Weinzierl, *Comput. Phys. Commun.* **167**, 177 (2005).
[16] H. Frellesvig, D. Tommasini, C. Wever, *J. High Energy Phys.* **1603**, 189 (2016).
[17] D.J.B. Jonathan, M. Borwein, D.M. Bradley, P. Lisonek, *Trans. Am. Math. Soc.* **353**, 907 (2001).
[18] J. Ablinger, J. Blümlein, C. Schneider, *J. Math. Phys.* **54**, 082301 (2013).

[19] C. Duhr, "Mathematical Aspects of Scattering Amplitudes" in: Theoretical Advanced Study Institute in Elementary Particle Physics: Journeys Through the Precision Frontier: Amplitudes for Colliders (TASI 2014), Boulder, Colorado, June 2–27, 2014.
[20] E. Panzer, *Comput. Phys. Commun.* **188**, 148 (2014).
[21] R. Bonciani *et al.*, *J. High Energy Phys.* **1508**, 108 (2015).
[22] M. Spira, A. Djouadi, P. Zerwas, *Phys. Lett. B* **276**, 350 (1992).
[23] T. Gehrmann, S. Guns, D. Kara, *J. High Energy Phys.* **1509**, 038 (2015).
[24] C. Meyer, *J. High Energy Phys.* **1704**, 6 (2017) [arXiv:1611.01087 [hep-ph]].
[25] M. Argeri *et al.*, *J. High Energy Phys.* **1403**, 082 (2014).
[26] R.N. Lee, *J. High Energy Phys.* **1504**, 108 (2015).
[27] Z. Bern, L.J. Dixon, D.C. Dunbar, D.A. Kosower, *Nucl. Phys. B* **435**, 59 (1995).
[28] R. Britto, F. Cachazo, B. Feng, *Nucl. Phys. B* **725**, 275 (2005).
[29] G. Ossola, C.G. Papadopoulos, R. Pittau, *Nucl. Phys. B* **763**, 147 (2007).
[30] D. Forde, *Phys. Rev. D* **75**, 125019 (2007).
[31] C.F. Berger *et al.*, *Phys. Rev. D* **78**, 036003 (2008).
[32] D.A. Kosower, K.J. Larsen, *Phys. Rev. D* **85**, 045017 (2012).
[33] S. Caron-Huot, K.J. Larsen, *J. High Energy Phys.* **1210**, 026 (2012).
[34] S. Badger, H. Frellesvig, Y. Zhang, *J. High Energy Phys.* **1204**, 055 (2012).
[35] S. Badger, H. Frellesvig, Y. Zhang, *J. High Energy Phys.* **1312**, 045 (2013).
[36] F. Cachazo, arXiv:0803.1988 [hep-th].
[37] N. Arkani-Hamed, J.L. Bourjaily, F. Cachazo, J. Trnka, *J. High Energy Phys.* **1206**, 125 (2012).
[38] N. Arkani-Hamed, J.L. Bourjaily, F. Cachazo, J. Trnka, *Phys. Rev. Lett.* **113**, 261603 (2014).
[39] H. Frellesvig, C.G. Papadopoulos, *J. High Energy Phys.* **1704**, 083 (2017).
[40] P.A. Baikov, *Phys. Lett. B* **385**, 404 (1996).
[41] P.A. Baikov, *Nucl. Instrum. Methods Phys. Res. A* **389**, 347 (1997).
[42] V.A. Smirnov, M. Steinhauser, *Nucl. Phys. B* **672**, 199 (2003).
[43] A.G. Grozin, *Int. J. Mod. Phys. A* **26**, 2807 (2011).
[44] S. Laporta, E. Remiddi, *Nucl. Phys. B* **704**, 349 (2005).
[45] M. Søgaard, Y. Zhang, *Phys. Rev. D* **91**, 081701 (2015).
[46] L. Adams, C. Bogner, S. Weinzierl, *J. Math. Phys.* **56**, 072303 (2015).
[47] M. Caffo, H. Czyz, S. Laporta, E. Remiddi, *Nuovo Cim. A* **111**, 365 (1998).
[48] L. Adams, S. Weinzierl, arXiv:1704.08895 [hep-ph].
[49] M. Hidding, F. Moriello, arXiv:1712.04441 [hep-ph].
[50] J. Broedel, C. Duhr, F. Dulat, L. Tancredi, arXiv:1712.07089 [hep-th].
[51] J. Broedel, C. Duhr, F. Dulat, L. Tancredi, arXiv:1712.07095 [hep-ph].
[52] K.J. Larsen, Y. Zhang, *Phys. Rev. D* **93**, 041701 (2016).

[53] A. Primo, L. Tancredi, *Nucl. Phys. B* **916**, 94 (2017) [arXiv:1610.08397 [hep-ph]].
[54] A. Georgoudis, K.J. Larsen, Y. Zhang, *Comput. Phys. Commun.* **221**, 203 (2017).
[55] J. Bosma, M. Sogaard, Y. Zhang, *J. High Energy Phys.* **1708**, 051 (2017).
[56] M. Zeng, *J. High Energy Phys.* **1706**, 121 (2017).
[57] A. Primo, L. Tancredi, *Nucl. Phys. B* **921**, 316 (2017).
[58] M. Harley, F. Moriello, R.M. Schabinger, *J. High Energy Phys.* **1706**, 049 (2017).
[59] K.-T. Chen, *Bull. Amer. Math. Soc.* **83**, 831 (1977).