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APPLICATION OF HYPER-SPHERICAL THREE-BODY VARIABLES TO LATTICE QCD DATA*

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Is the three-quark confinement potential better described by the Y-string or the Δ-string form? We have re-analysed the recent lattice QCD calculations and the older results using hyper-spherical three-body variables. The presently available lattice data do not give a conclusive answer to the above question. We briefly discuss sources of uncertainties in these calculations, the ways to reduce them, and how to avoid them.

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1. Introduction

The form of the three-heavy-quark potential in (lattice) QCD is not well-known. It is expected from general principles, see Refs. [1,2], that this potential contains the QCD Coulomb interaction and a confining part that rises linearly with increasing distances among the three quarks. The latter is not fully determined by this description, as there are infinitely many different homogeneous functions with degree of homogeneity +1. Dual resonance models, as well as the hadronic string ones, suggest that a gluonic string is formed among the quarks, which provides the (homogeneous) confining potential. Even within this narrower class of models, there are at least two distinct kinds of strings: the Δ-string and the Y-string. In the Y-string description, the potential depends on the sum of distances from the three

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quarks to the Fermat–Torricelli point of the system. In the Δ-string description, the potential goes as the perimeter (the sum of sides) of the triangle subtended by the three quarks[1].

This dilemma seems perfect for the lattice to resolve. Lattice QCD offers a method to calculate the three-quark potential *ab initio*. It must be kept in mind, however, that lattice calculations are numerical simulations that are statistical in nature, and are naturally limited by the lattice size, as well as by the available computer power. Therefore, lattice results are subject to systematic and statistical errors which must be estimated.

Two older calculations of the effective static three-quark potential on the lattice are the one by Alexandrou *et al.* [3], who proposed the Δ-string form, and the one by Takahashi *et al.* [4], who strongly claimed the Y-string. The recent calculation of the three-quark potential by Koma and Koma [5], on a larger (24^4) lattice, has re-ignited interest in this question. These authors did not test the potential for the O(2) dynamical symmetry, however, so their work cannot be conclusive with regard to the Y *versus* Δ dilemma.

The aim of this work is to use hyper-spherical coordinates to re-analyse both the Koma and the Takahashi data so as to see if it is possible on the basis of presently available lattice data to decide on the Y *versus* Δ dilemma. We shall show that a clear resolution is not possible, due to an unfortunate choice of triangle shapes.

2. Hyper-spherical coordinates

The three bodies' Cartesian coordinates can be expressed in terms of the centre-of-mass variable $\boldsymbol{R}_{\rm CM}$, and the two relative Jacobi vectors $\boldsymbol{\rho} = \frac{1}{\sqrt{2}}(\boldsymbol{x_1} - \boldsymbol{x_2})$, $\boldsymbol{\lambda} = \frac{1}{\sqrt{6}}(\boldsymbol{x_1} + \boldsymbol{x_2} - 2\boldsymbol{x_3})$ which obscure the permutation symmetry, however. The confining potential must be invariant under: (1) translations — the potential may depend only on the relative Jacobi vectors and not on $\boldsymbol{R}_{\rm CM} = \frac{1}{3}(\boldsymbol{x_1} + \boldsymbol{x_2} + \boldsymbol{x_3})$; (2) rotations — the potential can only depend on the scalar products of $\boldsymbol{\rho}$ and $\boldsymbol{\lambda}$; (3) Permutations of particle labels. There is one preferred set of hyper-angles that makes the permutation symmetry manifest: the hyper-angles $\phi = \arctan\left(\frac{2\boldsymbol{\rho}\cdot\boldsymbol{\lambda}}{\boldsymbol{\rho}^2 - \boldsymbol{\lambda}^2}\right)$ and $\alpha = \arccos\left(\frac{2(\boldsymbol{\rho}\times\boldsymbol{\lambda})}{\boldsymbol{\rho}^2 + \boldsymbol{\lambda}^2}\right)$ describe the shape of the triangle. The two hyper-angles determine a point on a unit-radius shape sphere. Figure 1 in Ref. [6] shows a view of the shape sphere from infinity above the North Pole. The hyper-radius $R = \sqrt{\boldsymbol{\rho}^2 + \boldsymbol{\lambda}^2}$ is the only variable with a dimension of length. As such, it determines the size of the triangle subtended by the three particles.

[1] Explicit formulae for the functional forms of the Y- and Δ-string potentials are given in Refs. [6–8]. Both potentials depend linearly on the "overall size" variable, the hyper-radius R of the three-quark system.

The lattice data from Refs. [4,5] were converted into permutation-adapted hyper-spherical coordinates by the above equations. We plotted those data points in the x–y projection of the shape sphere. Figure 1 includes all permutations of the three-body system. There are three lines that cross the center of the circle — representing isosceles triangles. There are also three lines representing the right-angled triangles — one such line is at $y = -0.5$. Equilateral triangles are located at the center of the circle. All collinear configurations lie on the perimeter of the circle.

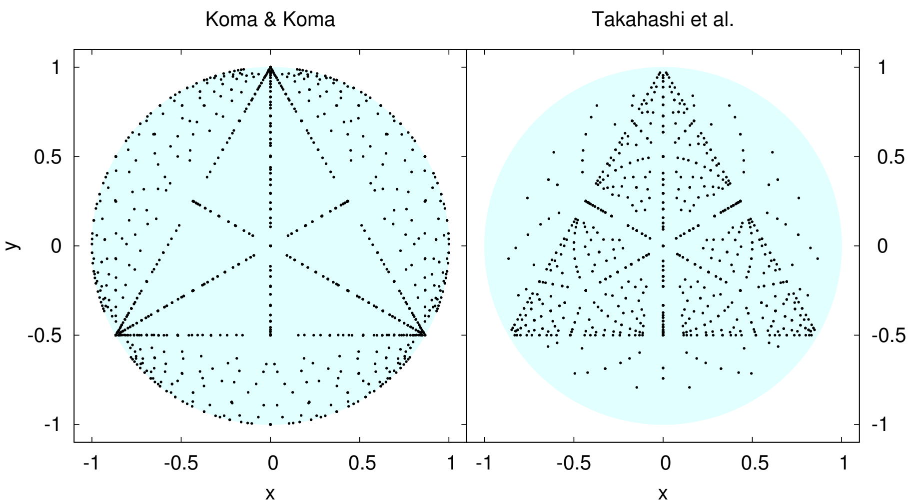

Fig. 1. Hyper-radius as a function of hyper-angles: for Koma and Koma, Ref. [5], data points (l.h.s. panel); for Takahashi, Ref. [4], data points (r.h.s. panel).

Note that the two data sets in Fig. 1 contain strikingly different shapes: whereas most of Ref. [5] points (triangle shapes) lie outside of the three straight lines in Fig. 1, defined by the right-triangle configurations, most of Ref. [4] points lie inside of this boundary. Moreover, most of Ref. [4] points that lie inside the right-triangle lines have low-hyper-radii. Therefore, this data set is ill-suited to address the Y *versus* Δ dilemma, as one needs high(er) values of the hyper-radius, so as to enhance the confining parts and suppress the Coulomb term.

3. Analysis of lattice data

Following standard lattice QCD treatises, Refs. [1, 2], we assume that the total three-quark potential V_{3q} has the form of

$$V_{3q}(\alpha, \phi, R) = -\frac{A(\alpha, \phi)}{R} + B(\alpha, \phi)R + C\,, \tag{1}$$

henceforth referred to as the Coulomb + linear potential Ansatz. The first term represents the sum of QCD Coulomb pairwise interactions, which is dominant at small values of the hyper-radius R. The second term represents the confinement potential, which is linear in R and dominant at large values of hyper-radius R, and the third term, C, is a constant. Here, $A(\phi, \alpha)$ is assumed to be the sum of pair-wise Coulomb terms, and $B(\phi, \alpha)$ is the unknown hyper-angular dependence of the linearly rising confining potential.

Our initial goal was to determine the unknown hyper-angular dependence, $B(\phi, \alpha)$, of the linearly rising confining part of the three-quark potential using the lattice data and the well-known hyper-angular and hyper-radial dependences of the two-body Coulomb term.

In order to minimize the influence of the Coulomb term and of the constant[2] C, we have only used lattice configurations that are either: (a) far away from the two-body collision points, where the Coulomb term rises to infinity; or (b) have large values of the hyper-radius R, where the Coulomb term is suppressed compared with the confining term. There are only two such sets of configurations in the Koma and the Takahashi data sets: *(i)* the isosceles; *(ii)* the right-angled triangles. We use them both.

(i) It can be seen in Fig. 2 (right) that for the isosceles triangle configurations ϕ =const. in the Koma and Koma data set, the $B(\alpha)$ values form a (more or less) continuous curve that passes between the Δ- and Y-string potentials' functional forms. This result supports the conclusion of Koma and Koma [5] that the three-quark potential is neither the Y- nor the Δ-string. The Takahashi data [4] are strongly scattered and often lie outside the region limited by the two functional forms.

(ii) The right-angled triangle configurations provide even less insight: The functional forms for the Y- and Δ-string potentials are very similar, differing mostly by a constant off-set, so the data do not allow a clear attribution. The Koma and Koma data points in Fig. 2 (left) follow a shape that could be attributed to either functional form. In the Takahashi data set, the points are too scattered to draw any conclusion.

Note, however, that at both: (1) the $x = 0, y = -0.5$ point in Fig. 2 (left); and (2) the $x = 0, y = 0$ point in Fig. 2 (right), there are several points that lie close to each other, and yet have different values of the potential V, corresponding to several triangles with identical shapes but with different sizes. Nominally, they should all have the same value of V — the differences signify a systematic error, which may stem from a number of sources.

[2] There is the possibility that the "constant" C is actually a function of the shape-sphere angles $C(\phi, \alpha)$.

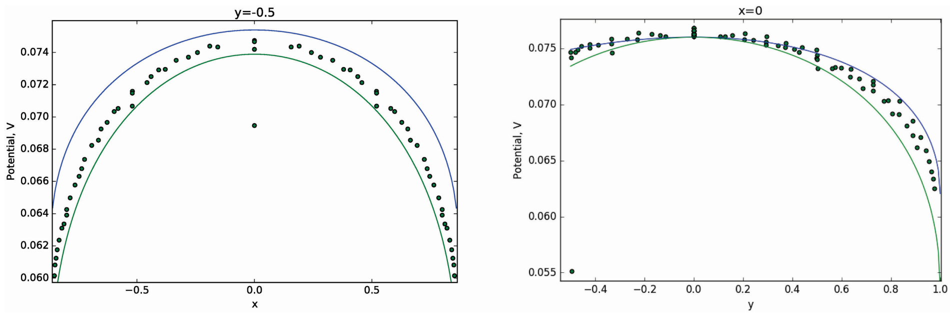

Fig. 2. (Colour on-line) Plot of the hyper-angular part of confining potential as a function of x or y for isosceles and right-angled triangle configurations in the Koma and Koma data set — the darker (blue) curves represent the Δ-string, and the lighter (green) curves are the Y-string predictions. Left: Right-angled triangles; Right: Isosceles triangles.

4. Discussion and conclusions

We have analysed the lattice QCD data from Refs. [4, 5] in terms of permutation-adapted hyper-spherical variables with a view towards resolving the Δ- *versus* Y-string dilemma. The analysis was done on two sets of configurations of three quarks that are common to both Refs. [4, 5] with the following results:

(1) The isosceles configurations in the Koma and Koma data [5] yield a confining potential that lies roughly half-way between the pure Y-string and the pure Δ-string. The Takahashi data [4], on the other hand, are too few and too scattered, as to allow any conclusion at all.

(2) Similarly, the right-angled triangle configurations in both Refs. [4, 5], are too few to provide a significant insight into the Δ- *versus* Y-string dilemma.

Why are these data so scattered? There are several possible sources of uncertainties (statistical and systematic errors). Here, we shall mention only one — the inadequacy of the assumed Ansatz (1). Koma and Koma [5] have noticed that the sum of pairwise Coulomb terms does not adequately describe the lattice data as the hyper-radius R decreases. Depending on the shape of the triangle, this discrepancy may amount up to 26% of the total Coulomb potential (see Eq. (34) in Sect. III. E of [5]). Therefore, a detailed study of both statistical and systematic errors must be completed, before any conclusions are drawn.

Finally, we note that the Y-string has an O(2) dynamical symmetry, in a major part of the shape space, see Refs. [6–8], which the Δ-string does not share. This dynamical O(2) symmetry is visible to the naked eye when using permutation-adapted hyper-spherical coordinates, see Fig. 2 in [8]. So, perhaps the best way to distinguish the Y-string from the Δ-string is to use this symmetry as a test. This calls for new lattice calculations, with: (1) a constant hyper-radius and different hyper-angles; (2) one of the hyper-angles is held constant while varying the other, so as to test the dynamical O(2) symmetry of the Y-string potential.

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