A FRESH LOOK AT THE (NON-)ABELIAN LANDAU–KHALATNIKOV–FRADKIN TRANSFORMATIONS *

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The Landau–Khalatnikov–Fradkin transformations (LKFTs) allow to interpolate n-point functions between different gauges. In this work, we offer a derivation for both Abelian and non-Abelian LKFT using gauge-invariant fields. Secondly, this subject is studied using a direct path integral formalism, finding full consistency.

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1. Introduction

In Quantum Chromodynamics (QCD), we start from basic fields, quarks, gluons and ghosts (in covariant gauges). However, due to the infrared enhancement of the strong coupling constant, perturbation theory does not suffice for describing strong interactions. A solution is given by non-perturbative QCD, requiring a different treatment of interaction between fields.

In the continuum formulation, gauge fixing is required to warrant computations, whatever non-perturbative scheme one has in mind. However, QCD remains a gauge theory, meaning physically observable quantities should not depend on what gauge is actually chosen to carry out the computation.

Secondly, most computations are limited to this Landau gauge, but other gauge choices are emerging. LKFT can now be used to study the relation between quantities studied in Landau gauge, and the same quantity in a newly chosen gauge. In practice, one often comes across some Ansatz, for instance, the photon–fermion vertex in QED. These Ansatze are fine-tuned to match observation, but this way they become gauge-specific but the corresponding LKFT can show how this vertex looks in a different gauge.

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Motivated by this, our ultimate goal is to study an LKFT for a general n-point quark–gluon correlator. In this work, we start by introducing gauge invariant fields, and show how these can be used to find both Abelian and non-Abelian LKFTs. Secondly, using the viewpoint of the path integral formalism, the gauge symmetries are fully exploited to find an alternative derivation of the LKFTs, finding the same results as in the first section.

2. LKFTs using gauge-invariant fields

2.1. A summary of the gauge invariant fields A^h, ψ^h

We start from the action [1, 2]

$$
\begin{aligned}
S &= S_{\mathrm{FP}} + S_f + S_h \,, \\
S_{\mathrm{FP}} &= \int \mathrm{d}^4x \left(\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \frac{\alpha}{2} b^a b^a + i b^a \partial_\mu A^a_\mu + \bar{c}^a \partial_\mu D^{ab}_\mu c^b \right) , \\
S_f &= \int \mathrm{d}^4x \left(\bar{\psi} \left(i\not{D} + m_f \right) \psi \right) , \\
S_h &= \int \mathrm{d}^4x \left(\tau^a \partial_\mu A^{h,a}_\mu + \frac{m^2}{2} A^{h,a}_\mu A^{h,a}_\mu + \bar{\eta}^a \partial_\mu D^{ab}_\mu (A^h) \eta^b \right) ,
\end{aligned}
\tag{1}
$$

where A^h_μ is defined through

$$
A^h_\mu = h^\dagger A_\mu h + \frac{i}{g} h^\dagger \partial_\mu h \quad \text{with} \quad h = e^{ig\phi^a T^a} \,, \tag{2}
$$

with T^a the adjoint generators of $\mathrm{SU}(N)$. As it is apparent, the action S_h contains a new field ϕ^a, besides the Lagrange multiplier τ^a as well as the additional ghost fields $(\bar{\eta}^a,\, \eta^a)$[1]. All these fields belong to the adjoint representation.

The gauge invariance of A^h_μ can be nicely appreciated from the transformation laws

$$
h \to U^\dagger h \,, \qquad h^\dagger \to h^\dagger U \,, \qquad A_\mu \to A^U_\mu = U^\dagger A_\mu U + \frac{i}{g} U^\dagger \partial_\mu U \,, \tag{3}
$$

with U a generic local $\mathrm{SU}(N)$ transformation. Looking at the equations of motion for the τ field, it is evident that the field A^h is transverse, $\partial_\mu A^h_\mu = 0$.

[1] As underlined in [2], the additional ghosts $(\bar{\eta}^a, \eta^a)$ are needed to take into account the Jacobian arising from integration over the Lagrange multiplier τ^a, which gives rise to a delta function of the type $\delta(\partial_\mu A^{h,a}_\mu)$.

This constraint can be solved power-by-power, the result is given by

$$
\begin{aligned}
A^h_\mu \;=\; & A_\mu - \frac{\partial_\mu}{\partial^2}\partial A + ig\left[A_\mu, \frac{1}{\partial^2}\partial A\right] + \frac{ig}{2}\left[\frac{1}{\partial^2}\partial A, \partial_\mu \frac{1}{\partial^2}\partial A\right] \\
& + ig\frac{\partial_\mu}{\partial^2}\left[\frac{\partial_\nu}{\partial^2}\partial A, A_\nu\right] + i\frac{g}{2}\frac{\partial_\mu}{\partial^2}\left[\frac{\partial A}{\partial^2}, \partial A\right] + \mathcal{O}\left(A^3\right) . \qquad (4)
\end{aligned}
$$

Furthermore, it has been proven that this construction is BRST-invariant and renormalizable, for a deeper insight on this, we refer to Ref. [1].

2.2. Derivation of the (non-)Abelian LKFTs

In the Abelian limit, expansion (4) simplyfies to $A^h_\mu = A_\mu - \partial_\mu \phi$, which can be used to obtain the expectation value

$$
\left\langle A^h_\mu(p) A^h_\nu(-p) \right\rangle_\alpha = \left\langle A_\mu(p) A_\nu(-p) \right\rangle_\alpha - \alpha \frac{p_\mu p_\nu}{p^4} , \qquad (5)
$$

or, specifying to the Landau gauge ($\alpha = 0$), $\langle A^h_\mu A^h_\nu \rangle_{\alpha=0} = \langle A_\mu A_\nu \rangle_{\alpha=0}$.

It is worth now to remind that the transverse field A^h_μ is gauge-invariant or, equivalently, BRST-invariant, see [1,2]. From this important feature, it follows that the correlation function $\langle A^h_\mu(p) A^h_\nu(-p) \rangle_\alpha$ is BRST-invariant as well. As such, it does not depend on the gauge parameter α [1,2], $\langle A^h_\mu A^h_\nu \rangle_\alpha = \langle A^h_\mu A^h_\nu \rangle_{\alpha=0}$, and we find

$$
\left\langle A_\mu(p) A_\nu(-p) \right\rangle_\alpha = \left\langle A_\mu(p) A_\nu(-p) \right\rangle_{\alpha=0} + \alpha \frac{p_\mu p_\nu}{p^4} . \qquad (6)
$$

Said otherwise, we simply recover the LKFT for the photon. Of course, this result can also be easily derived using the underlying BRST invariance of the theory, which ensures that the longitudinal component of the gluon propagator does not receive any quantum correction, being given by its tree-level approximation.

2.3. General LKFTs

Similar to the construction of A^h, gauge-invariant fermion fields can be constructed as well. Consider $\psi^h = h^\dagger \psi$, with h being still defined via Eq. (2), the gauge invariance of ψ^h can easily be verified. The combination of these invariant fields can be used to study general n-point functions

$$
\left\langle A^h_\mu \ldots \psi^h \ldots \bar{\psi}^h \ldots \right\rangle_\alpha = \left\langle A^h_\mu \ldots \psi^h \ldots \bar{\psi}^h \ldots \right\rangle_{\alpha'} . \qquad (7)
$$

Specifically, this can be used to obtain the LKFT for the gluon propagator, using the full expansion Eq. (4).

3. LKFTs using the path integral and gauge transformations

In what follows, we will refresh the direct path integral derivation of the Abelian LKFT, which is a kind of rewriting of the original argument provided in [3, 4] in a more modern language. As an expansion, we will generalize this derivation to the non-Abelian case, at the cost of adding several complications of course.

Consider for now the QED action

$$S = \int \mathrm{d}[4]x \left(\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \bar{\psi} \not{D} \psi + ib \partial_\mu A_\mu + \frac{\alpha}{2} b^2 + \bar{c} \partial^2 c + \bar{J}_\psi \psi + \bar{\psi} J_{\bar{\psi}} \right) , \tag{8}$$

where we included sources for ψ and $\bar{\psi}$ to define the generating functional of Green's functions, $Z(J)$, via the path integral $Z(J) = \int [\mathrm{d}\mu]\ e^{-S}$.

Next, we transform the path integral variables A, ψ, and $\bar{\psi}$ using the gauge transformation

$$U = e^{ie\phi} , \qquad A_\mu \to A'_\mu = A_\mu - \partial_\mu \phi , \qquad \psi \to \psi' = U^\dagger \psi , \tag{9}$$

and we select

$$\phi = -X \frac{1}{\partial^2} \partial_\mu A_\mu , \tag{10}$$

where the constant X can still be chosen appropriately, see later. The specific choice of ϕ can easily be appreciated looking at the transformation of

$$\partial_\mu A_\mu \to \partial_\mu A'_\mu = (1 + X) \partial_\mu A_\mu \tag{11}$$

resulting in a rescaling of the Lagrange multiplier b and gauge paramater α to keep the action invariant

$$b \to b' = \frac{1}{1+X} b , \tag{12}$$

$$\alpha \to \alpha' = (1+X)^2 \alpha . \tag{13}$$

The action, up to its source part, is transformed into itself, except that the gauge parameter α gets replaced by α'. Importantly, also the source terms vary, which will result in the correlator in the new gauge.

It is consequently found that the ϕ-propagator has the expected form, contracting the propagator $\langle \phi\phi \rangle$ in the α' gauge, we obtain $\langle \phi(p)\phi(-p) \rangle_{\alpha'} = -X^2 \alpha \frac{1}{p^4}$, in accordance with Ref. [4]. For the formal derivation of this propagator, we refer to Ref. [5].

Starting from any gauge α, if we take the limit $X \to -1$, we arrive at the Landau gauge $\alpha' = 0$, while the ϕ-propagator remains proportional to $\frac{1}{p^4}$. This rather singular behaviour is fundamental to correctly transform the longitudinal projection of the gluon propagator. We remind here that the latter projection is uniquely fixed by means of the underlying Ward identities, *i.e.* the BRST invariance as well as other additional Ward identities defining the class of linear covariant gauges at the quantum level, see, for instance, Ref. [1].

We can also investigate the photon propagator. Therefore, we add the term $J_\mu A_\mu$ to the action, which will transform to $J_\mu\left(A'_\mu - \frac{X}{1+X}\frac{1}{\partial^2}\partial_\mu\partial_\nu A'_\nu\right)$ after the chosen gauge transformation. The photon propagator can be found using

$$\langle A_\mu(x)A_\nu(y)\rangle_\alpha = \frac{\delta^2 Z_\alpha}{\delta J_\nu(y)\delta J_\mu(x)} \tag{14}$$

in the original gauge, or in the new gauge by replacing $\alpha \to \alpha'$.

From this, we find for the photon propagator

$$\langle A_\mu(p)A_\nu(-p)\rangle_\alpha = \left\langle A'_\mu(p)A'_\nu(-p)\right\rangle_{\alpha'} - \left(\alpha' - \alpha\right)\frac{p_\mu p_\nu}{p^4}\,. \tag{15}$$

Clearly, Eq. (15) expresses that only the longitudinal part of the photon propagator is affected by shifting $\alpha \to \alpha'$, exactly the same result as Eq. (6).

We now wish to generalize the foregoing path integral derivation of the LKFTs to a non-Abelian gauge theory supplemented with fermion matter.

We must first establish a general SU(N) transformation with matrix U for all fields, while maintaining the property that $\partial_\mu A'_\mu = (1+X)\partial_\mu A_\mu$. This is a necessary requirement, as it will precisely allow for the rescaling of the Lagrange multiplier b, and thereby for that of the gauge parameter α.

The gauge and matter fields transform as

$$A_\mu \to A'_\mu \;=\; U^\dagger A_\mu U + \frac{i}{g}U^\dagger\partial_\mu U\,, \tag{16}$$

$$\psi \to \psi' \;=\; U^\dagger\psi \quad \text{with} \quad U = e^{ig\phi}\,. \tag{17}$$

The requirement $\partial_\mu A'_\mu = (1+X)\partial_\mu A_\mu$ can be solved order-by-order to the rotation angle ϕ, the gauge transformed field A'_μ as a function of the original A_μ is found to be

$$\begin{aligned} A'_\mu &= A_\mu + X\frac{\partial_\mu\partial A}{\partial^2} - igX\frac{\partial_\mu}{\partial^2}\left[\frac{\partial_\nu\partial A}{\partial^2}, A_\nu\right] - igX\frac{\partial_\mu}{\partial^2}\left[\frac{\partial A}{\partial^2}, \partial A\right] \\ &-\frac{igX^2}{2}\frac{\partial_\mu}{\partial^2}\left[\frac{\partial A}{\partial^2}, \partial A\right] + igX\left[\frac{\partial A}{\partial^2}, A_\mu\right] + \frac{igX^2}{2}\left[\frac{\partial A}{\partial^2}, \frac{\partial_\mu\partial A}{\partial^2}\right] + \mathcal{O}\left(A^3\right)\,. \end{aligned} \tag{18}$$

Note that for the Landau gauge, $X = -1$, this expression coincides with the gauge-invariant transversal field A^h_μ, see Eq. (2) or Ref. [1]. In general, A'_μ will not be transversal though.

In order that $Z_\alpha = Z_{\alpha'}$, we used that the action remains invariant. However, one also has to show that the integration measure does not change under the chosen gauge transformation. The interested reader is redirected to Ref. [5].

4. Conclusion and outlook

We have employed the gauge-invariant fields A^h_μ and ψ^h to provide an alternative way to derive the LKFTs for general n-point correlators. This derivation was first performed for the Abelian LKFT for the photon and fermion fields. It reproduced the correct relations as already known from the literature. The extension to non-Abelian theories was then presented. To our knowledge, this is the first time in which the non-Abelian LKFTs have been derived for arbitrary n-point correlators without any approximation.

To lend further credit to the validity of our non-Abelian LKFTs, we also presented an independent derivation of the LKFTs, from the viewpoint of the path integral formalism, leading to exactly the same transformations.

As a second proof of concept, we will study the LKFT for the gluon propagator, which in turn can be compared to lattice QCD predictions to verify the construction using gauge-invariant fields. Secondly, it would be interesting to link these two viewpoint with the Nielsen identities, linking the n-point correlators.

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REFERENCES

[1] M.A.L. Capri *et al.*, *Phys. Rev. D* **94**, 065009 (2016) [`arXiv:1606.06601 [hep-th]`].

[2] M.A.L. Capri, D. Fiorentini, A.D. Pereira, S.P. Sorella, *Phys. Rev. D* **96**, 054022 (2017) [`arXiv:1708.01543 [hep-th]`].

[3] L.D. Landau, I.M. Khalatnikov, *Sov. Phys. JETP* **2**, 69 (1956) [*Zh. Eksp. Teor. Fiz.* **29**, 89 (1955)].

[4] E.S. Fradkin, *Sov. Phys. JETP* **2**, 361 (1956) [*Zh. Eksp. Teor. Fiz.* **29**, 258 (1955)].

[5] T. De Meerleer *et al.*, *Phys. Rev. D* **97**, 074017 (2018) [`arXiv:1801.01703 [hep-th]`].