EXPLORATIONS OF TERNARY CELLULAR AUTOMATA AND TERNARY DENSITY CLASSIFICATION PROBLEMS*

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While binary nearest-neighbour cellar automata (CA) have been studied in detail and from many different angles, the same cannot be said about ternary (three-state) CA rules. We present some results of our explorations of a small subset of the vast space of ternary rules, namely rules possessing additive invariants. We first enumerate rules with four different additive invariants, and then we investigate if any of them could be used to construct a two-rule solution of generalized density classification problem (DCP). We show that neither simple nor absolute classification is possible with a pair of ternary rules, where the first rule is all-conserving and the second one is reducible to two states. Similar negative result holds for another version of DCP we propose: symmetric interval-wise DCP. Finally, we show an example of a pair of rules which solve non-symmetric interval-wise DCP for initial configurations containing at least one zero.

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1. Introduction

In both the theory and applications of cellular automata, a lot of effort has been invested in studying binary rules. Given the importance of binary logic and binary arithmetic in todays computing, this is of course quite understandable.

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Nevertheless, the next possible arithmetical and logical systems in terms of the number of allowed states, namely the ternary arithmetic and ternary logic, enjoyed some (if only limited) popularity in the past. To give concrete examples, the wooden calculating machine [1] built by Thomas Fowler in 1840 operated in balanced ternary arithmetic, using three digits -1, 0 and 1. In the early 20$^{\text{th}}$ century, Polish mathematician Jan Łukasiewicz invented and formalized a three-valued logic [2], and many followers developed his ideas in subsequent years. In the second half of the 20$^{\text{th}}$ century, experimental computers based on ternary arithmetic were constructed, beginning with the most famous one, the *Setun* computer developed in 1958 at the Moscow State University [3]. In 1980s, ternary CMOS memory chips (ROM) were built at Queens University in Canada. In the most recent times, the possibility of quantum computing based on qtrits instead qbits has been suggested [4].

All of the above indicates that ternary arithmetic and ternary logic are still interesting areas to explore, and that they may have some application potential. This inspired us to probe into the huge space of ternary cellular automata, with the hope of finding some interesting rules, possibly applicable to solving computational problems such as the density classification problem.

Of course, even if one considers only nearest-neighbour ternary cellular automata, the number of possible rules is huge, $3^{3^3} \approx 7.63 \times 10^{12}$. There is no hope to study all of them systematically, as it has been done in the case of elementary (binary nearest-neighbour) cellular automata, which form a small set of 256 rules. For this reason, we have decided to take a closer look at only a small subset of ternary rules, namely those which obey simple conservation laws, hoping to find some which would be useful in constructing solutions of computational problems similar to classical density classification problem (see [5] for review and references).

The paper is organized as follows. First, we enumerate ternary nearest-neighbour rules with various additive invariants. Then we investigate if any of the rules found could be used to solve a generalization of the density classification problem to 3 states.

Let us define some basic concepts first. We will be mostly concerned with the space $S = \{0, 1, 2\}^L$ of finite-length configurations of length L . We will impose periodic boundary conditions on configurations $\boldsymbol{x} \in S$, so that for $\boldsymbol{x} = (x_0, x_1, \ldots x_{L-1})$, the index i in x_i is to be always taken modulo L, *i.e.*, $i \in \mathbb{Z}/L$. The *local function* (also called *local transition function* or *local rule*) of a ternary nearest-neighbour cellular automaton will be a function $f : \{0, 1, 2\}^3 \to \{0, 1, 2\}$, while the corresponding global function $F : S \to S$ will be defined as

$$(F(\boldsymbol{x}))_i = f(x_{i-1}, x_i, x_{i+1}) \tag{1}$$

for all $i \in \mathbb{Z}/L$.

2. Conservation laws

Interesting classes of ternary rules are those with additive invariants. Let $\Psi(x)$ be some function of $x \in \{0,1,2\}$. If, for any periodic configuration $\boldsymbol{x}$ of length (period) L we have $\sum_{i=0}^{L-1} \Psi(x_i) = \sum_{i=0}^{L-1} \Psi((F(x))_i)$, then we say that F conserves Ψ.

According to a well-known theorem of Hattori and Takesue [6], Ψ is conserved if and only if for all $x_1, x_2, x_3 \in \{0,1,2\}$ we have

$$\begin{aligned} &\Psi(f(x_0,x_1,x_2)) - \Psi(x_0) = \Psi(f(p,p,x_1)) \\ &+\Psi(f(p,x_1,x_2)) - \Psi(f(p,p,x_0)) - \Psi(f(p,x_0,x_1))\,, \end{aligned} \tag{2}$$

where p is an arbitrarily chosen element of the alphabet $\{0,1,2\}$. For the sake of convenience, we will use $p = 0$ in all what follows.

Equation (2) can be understood as a sort of discrete form of the continuity equation $\partial\rho/\partial t = -\partial j/\partial x$ describing transport of some quantity with density ρ, where j is the flux of this quantity. It can be transformed to another, equivalent form, where on the left-hand side, one has the local change of Ψ in one time step, analogous to $\partial\rho/\partial t$, and on the right-hand side, one has an expression analogous to the spatial derivative of the flux of Ψ (see [6] for further details).

The Hattori–Takesue theorem makes it quite easy to check if a given ternary rule f conserves Ψ — all one needs to do is to verify the above condition for all $3^3 = 27$ possible sets of values of $x_1, x_2, x_3 \in \{0,1,2\}$.

Examples of some possible choices of Ψ are:

(a) $\Psi_0(x) = \frac{1}{2}(1-x)(2-x)$, number of cells in state 0 is conserved;

(b) $\Psi_1(x) = x(2-x)$, number of cells in state 1 is conserved;

(c) $\Psi_2(x) = \frac{1}{2}x(x-1)$, number of cells in state 2 is conserved;

(d) $\Psi_s(x) = x$, sum of cell values is conserved (so-called *number-conserving rule*).

Note that conservation of any two of the above implies conservation of the remaining two. Rules which conserve all (a)–(d) will be called *all-conserving*.

3. Enumeration

We will first enumerate ternary rules conserving various Ψs. The following proposition provides a summary of such enumeration.

Proposition 1 *Among nearest-neighbour cellular automata with three states, there is exactly*

(a) $9 \times 2^{18} = 2359296$ *rules conserving* Ψ_0 *(the same applies to* Ψ_1 *and* Ψ_2*);*

(b) 144 rules conserving Ψ_s;

(c) 15 all-conserving rules.

Proof: Claim (b) is a known result [7], thus we will prove only (a) and (c). Let $f : \{0,1,2\}^2 \to \{0,1,2\}$ be a local function of a CA conserving 1s. According to the Hattori–Takesue theorem, it must satisfy

$$\begin{aligned} &\Psi_1(f(x_0,x_1,x_2)) - \Psi_1(x_0) = \Psi_1(f(0,0,x_1)) \\ &+\Psi_1(f(0,x_1,x_2)) - \Psi_1(f(0,0,x_0)) - \Psi_1(f(0,x_0,x_1)) \end{aligned} \tag{3}$$

for all $x_1, x_2, x_3 \in \{0,1,2\}$, where $\Psi_1(x) = x(2-x)$.

When values of arguments of f are restricted to only 0s and 1s, f must return outputs compatible with one of the five number-conserving elementary cellular automata (ECA) rules with Wolfram numbers 184, 226, 170, 240 or 204, where compatibility is defined as follows. We say that ternary rule f is *compatible* with binary rule g if, for $x_1, x_2, x_3 \in \{0,1\}$,

$$f(x_1,x_2,x_3) = \begin{cases} 1 & \text{if } g(x_1,x_2,x_3) = 1\,, \\ 0 \text{ or } 2 & \text{if } g(x_1,x_2,x_3) = 0\,. \end{cases} \tag{4}$$

In the above, we have "0 or 2" because we only want to preserve the number of 1s.

We will first enumerate rules which, when their arguments are restricted to 0s and 1s, are compatible with rule 184. Let us define, for $x_1, x_2, x_3 \in \{0,1,2\}$,

$$a_{9x_0+3x_1+x_2} = f(x_0,x_1,x_2)\,. \tag{5}$$

For rules compatible with rule 184, we must have

$$\begin{aligned} a_0 &= f(0,0,0) = 0 \text{ or } 2\,, \\ a_1 &= f(0,0,1) = 0 \text{ or } 2\,, \\ a_3 &= f(0,1,0) = 0 \text{ or } 2\,, \\ a_4 &= f(0,1,1) = 1\,, \\ a_9 &= f(1,0,0) = 1\,, \\ a_{10} &= f(1,0,1) = 1\,, \\ a_{12} &= f(1,1,0) = 0 \text{ or } 2\,, \\ a_{13} &= f(1,1,1) = 1\,. \end{aligned}$$

Note that the above implies $\Psi_1(a_0) = \Psi_1(a_1) = \Psi_1(a_3) = \Psi_1(a_{12}) = 0$. The remaining a_i must satisfy 27 conditions obtained from Eq. (3). All of them are listed below in two columns, including those which reduce to identities

$$0 = 0\,, \qquad \Psi_1(a_{14}) - 1 = -1 + \Psi_1(a_5)\,,$$
$$0 = 0\,, \qquad \Psi_1(a_{15}) - 1 = \Psi_1(a_2) + \Psi_1(a_6) - \Psi_1(a_5)\,,$$
$$\Psi_1(a_2) = \Psi_1(a_2)\,, \qquad \Psi_1(a_{16}) - 1 = \Psi_1(a_2) + \Psi_1(a_7) - \Psi_1(a_5)\,,$$
$$0 = 0\,, \qquad \Psi_1(a_{17}) - 1 = \Psi_1(a_2) + \Psi_1(a_8) - \Psi_1(a_5)\,,$$
$$1 = 1\,, \qquad \Psi_1(a_{18}) = -\Psi_1(a_2) - \Psi_1(a_6)\,,$$
$$\Psi_1(a_5) = \Psi_1(a_5)\,, \qquad \Psi_1(a_{19}) = -\Psi_1(a_2) - \Psi_1(a_6)\,,$$
$$\Psi_1(a_6) = \Psi_1(a_6)\,, \qquad \Psi_1(a_{20}) = -\Psi_1(a_6)\,,$$
$$\Psi_1(a_7) = \Psi_1(a_7)\,, \qquad \Psi_1(a_{21}) = -\Psi_1(a_2) - \Psi_1(a_7)\,,$$
$$\Psi_1(a_8) = \Psi_1(a_8)\,, \qquad \Psi_1(a_{22}) = 1 - \Psi_1(a_2) - \Psi_1(a_7)\,,$$
$$0 = 0\,, \qquad \Psi_1(a_{23}) = \Psi_1(a_5) - \Psi_1(a_2) - \Psi_1(a_7)\,,$$
$$0 = 0\,, \qquad \Psi_1(a_{24}) = \Psi_1(a_6) - \Psi_1(a_8)\,,$$
$$\Psi_1(a_{11}) - 1 = \Psi_1(a_2)\,, \qquad \Psi_1(a_{25}) = \Psi_1(a_7) - \Psi_1(a_8)\,,$$
$$-1 = -1\,, \qquad \Psi_1(a_{26}) = 0\,.$$
$$0 = 0\,,$$

From the above, if we discard all equations which are identically true and introduce variables $b_i = \Psi_1(a_i)$, we obtain

$$b_{11} - 1 = b_2\,, \qquad b_{20} = -b_6\,,$$
$$b_{14} - 1 = -1 + b_5\,, \qquad b_{21} = -b_2 - b_7\,,$$
$$b_{15} - 1 = b_2 + b_6 - b_5\,, \qquad b_{22} = 1 - b_2 - b_7\,,$$
$$b_{16} - 1 = b_2 + b_7 - b_5\,, \qquad b_{23} = b_5 - b_2 - b_7\,,$$
$$b_{17} - 1 = b_2 + b_8 - b_5\,, \qquad b_{24} = b_6 - b_8\,,$$
$$b_{18} = -b_2 - b_6\,, \qquad b_{25} = b_7 - b_8\,,$$
$$b_{19} = -b_2 - b_6\,, \qquad b_{26} = 0\,.$$

This is a linear system of 14 equations with 19 unknowns. We can, therefore, solve this system for $b_{11}, b_{14}, b_{15}, \ldots, b_{26}$ in terms of b_2, b_5, b_6, b_7, b_8, obtaining

$$b_{11} = 1 + b_2\,, \qquad b_{20} = -b_6\,,$$
$$b_{14} = b_5\,, \qquad b_{21} = -b_2 - b_7\,,$$
$$b_{15} = -b_5 + 1 + b_2 + b_6\,, \qquad b_{22} = 1 - b_2 - b_7\,,$$
$$b_{16} = -b_5 + 1 + b_2 + b_7\,, \qquad b_{23} = b_5 - b_2 - b_7\,,$$
$$b_{17} = -b_5 + 1 + b_2 + b_8\,, \qquad b_{24} = b_6 - b_8\,,$$
$$b_{18} = -b_2 - b_6\,, \qquad b_{25} = b_7 - b_8\,,$$
$$b_{19} = -b_2 - b_6\,, \qquad b_{26} = 0\,.$$

Recall that the only allowed values of b_i are 0 or 1, thus we must have $b_2 = 0$. By the same token, we also need $b_7 = 0$, $b_8 = 0$, and $b_6 = 0$. This leaves

$$
\begin{aligned}
b_{11} &= 1\,, & b_{20} &= 0\,,\\
b_{14} &= b_5\,, & b_{21} &= 0\,,\\
b_{15} &= -b_5 + 1\,, & b_{22} &= 1\,,\\
b_{16} &= -b_5 + 1\,, & b_{23} &= b_5\,,\\
b_{17} &= -b_5 + 1\,, & b_{24} &= 0\,,\\
b_{18} &= 0\,, & b_{25} &= 0\,,\\
b_{19} &= 0\,, & b_{26} &= 0\,,
\end{aligned}
$$

meaning that we have only one parameter left, b_5. Obviously, $b_5 \in \{0,1\}$, thus we obtain two solutions of our original system. The first corresponds to

$$
\begin{aligned}
b_i &= 0 \text{ for } \; i \in \{2,5,6,7,8,14,18,19,20,21,23,24,25,26\}\,,\\
b_i &= 1 \text{ for } \; i \in \{11,15,16,17,22\}\,,
\end{aligned}
$$

and the second to

$$
\begin{aligned}
b_i &= 0 \text{ for } \; i \in \{2,6,7,8,15,16,17,18,19,20,21,24,25,26\}\,,\\
b_i &= 1 \text{ for } \; i \in \{5,11,4,22,23\}\,.
\end{aligned}
$$

In both the above solutions, the number of b_is with zero values is exactly 14. Each $b_i = 0$ admits two corresponding values of a_i, $a_i = 0$ or $a_i = 2$, because we defined $b_i = \Psi_1(a_i) = a_i(2 - a_i)$. Moreover, values of 4 parameters a_0, a_1, a_2, and a_{12} also admit two possible values, 0 or 2. This means that each of these two solutions corresponds to 2^{18} possible sets $a_0, a_2, \ldots, a_{26}$ satisfying 27 Hattori–Takesue conditions. We thus have total 2^{19} CA rules conserving 1s and compatible with rule 184.

For rules compatible with ECA 226, the calculations are almost identical, yielding also 2^{19} rules. For rules compatible with ECA 170 or 240, by using similar reasoning as above, we obtain in both cases 2^{18} rules. For rules compatible with ECA 204, the total number of rules turns out to be 3×2^{18}, again by a similar reasoning (omitted here).

The total number of rules conserving 1s is, therefore, $2^{19} + 2^{19} + 2^{18} + 2^{18} + 3 \times 2^{18} = 9 \times 2^{18}$, as claimed in (a).

For part (c), we took the advantage of the fact that every all-conserving rule must also be number-conserving. One can, therefore, simply test all 144 number-conserving rules for conservation of the number of 0s (the numbers of 1s and 2s will then be automatically conserved, so they do not even need to be checked). We carried out his procedure and found that 15 rules shown in Table I are the only all-conserving ternary rules.

4. Classification problems

The following two-rule solution of the density classification problem is well-known [8].

Proposition 2 (H.F. 1997) *Let* $\boldsymbol{x}$ *be a periodic binary configuration of length* L *and density* $\rho = \frac{1}{L}\sum_{i=0}^{L-1} x_i$*, and let* $n = \lfloor (L-2)/2 \rfloor$*,* $m = \lfloor (L-1)/2 \rfloor$*. Then*

— $F_{232}^{m}(F_{184}^{n}(\boldsymbol{x})) = 0^L$ *if* $\rho < 1/2$,

— $F_{232}^{m}(F_{184}^{n}(\boldsymbol{x})) = 1^L$ *if* $\rho > 1/2$,

— $F_{232}^{m}(F_{184}^{n}(\boldsymbol{x})) = \ldots 01010101 \ldots$ *if* $\rho = 1/2$.

In the above, F_{184} and F_{232} are global functions of elementary cellular automata 184 and 232, and 0^L (1^L) denotes a string of length L of all zeros (all ones).

We say that the pair of rules (184, 232) classifies densities, or solves the *density classification problem* (DCP), because iterating rule 184 sufficient number of times followed by analogous iteration of rule 232 produces homogeneous string of all zeros if initially we had more zeros than ones, and homogeneous string of all ones if we had more ones than zeros at the beginning. It is worth noting that the above two-rule solution of DCP has been proposed because single-rule solution of this problem does not exist [9, 10].

There are three obvious ways to generalize the density classification problem to 3 states (or more).

— *0-majority:* in the final configuration, all sites are to be in state 0 if there are more zeros than other symbols in the initial configuration. If there are more non-zero symbols than zeros in the initial configuration, then in the final state all sites are to be in state 1;

— *simple majority:* in the final configuration all sites are to be in state k if symbols k form the majority in the initial configuration;

— *absolute majority:* in the final configuration all sites are to be in state k if symbols k form the absolute majority in the initial configuration.

Note that all three are compatible with the binary DCP, meaning that when the initial configuration contains only 0s and 1s, they reduce to the binary DCP.

Since the binary DCP has no single-rule solution, the same applies to all three generalized problems as well. But are there any two-rule solutions for these problems? In 1999, Chau *et al.* [11] constructed two-rule solution to n-ary simple majority problem. Their solution, however, uses rules of rather large neighbourhood size. For ternary rules, it requires that the first rule has neighbourhood radius of at least 45. Despite the fact that Chau's solution uses some very interesting ideas, we will not discuss it here, as we wish to focus on nearest-neighbour rules only. Along the same vein, since we restrict our attention to ternary rules only, we will not be concerned with solutions of density classification problems which require very large number states, such as, for example, the work of Briceño *et al.* [12].

Since in the known two-rule solution of the binary DCP the first rule conserves the number of zeros (and ones), it is reasonable to expect that for ternary rules, the first rule of the solution should also conserve the relevant quantity. For 0-majority problem, the fist rule should thus be Ψ_0-conserving, and for the simple or absolute majority problem, it should be all-conserving. Since we have enumerated various rules with invariants, we can search among the relevant rules for a possible candidate for the first rule of the solution.

For the 0-majority problem, we do not even have to search among the 9×2^{18} Ψ_0-conserving rules, because the solution is trivial to construct. Let

$$f(x_1, x_2, x_3) = f_{184}(\phi(x_1), \phi(x_2), \phi(x_3)) \quad \text{for all} \quad x_1, x_2, x_3 \in \{0, 1, 2\}\,, \tag{6}$$

where we define $\phi(0) = 0$, $\phi(1) = 1$, $\phi(2) = 1$ and where f_{184} is the local rule ECA 184. The pair f and ECA 232 solve 0-majority DCP in the same fashion as rules 184 and 232 solve the binary DCP. Obviously, one could also define two other variants of the 0-majority problem (1-majority and 2-majority), and construct their two-rule solutions in an analogous way.

5. Simple and absolute density classification

For the remaining two versions of DCP, we have not found any pair of ternary rules solving these versions. We strongly suspect that such a pair does not exist, although we were able to prove only a partial non-existence result, to be presented below.

Let us define, for a given ternary rule f, three functions

$$\begin{aligned} f|_{01}(x_1, x_2, x_3) &= f(x_1, x_2, x_3)\,, \\ f|_{02}(x_1, x_2, x_3) &= f(2x_1, 2x_2, 2x_3)/2\,, \\ f|_{12}(x_1, x_2, x_3) &= f(x_1 + 1, x_2 + 1, x_3 + 1) - 1\,, \end{aligned} \tag{7}$$

where $x_1, x_2, x_3 \in \{0, 1\}$. If any of these functions returns only values 0 or 1 for all $x_1, x_2, x_3 \in \{0, 1\}$, we will say that the relevant *binary reduction* of f exists. If all three binary reductions exist, we will call f *reducible to two states.*

Proposition 3 *There exists no pair of nearest-neighbour ternary rules solving the simple (or absolute) majority density classification task such that the first rule of the pair is all-conserving and the second rule is reducible to two states.*

Proof: Suppose that there exists such a pair of ternary rules solving the simple majority density classification, and the first rule f in this pair is all-conserving, while the second one is reducible to two states. Table I shows Wolfram numbers of binary projections for all 15 all-conserving rules. One can clearly see that the first rule, when restricted to binary configurations, behaves as one of the five rules among 204 (identity), 170 (left shift), 240 (right shift), 184, and 226. Since identity and shifts do not change the arrangement of symbols, f behaving as rule 204, 170 or 240 on binary configurations would require that the second rule performed the entire task of the density classification on its own (f would do nothing). We assumed reducibility of the second rule, thus its reduction to $\{0,1\}$ would have to be a solution of the binary classification problem. This, however is impossible — single-rule solution of the binary classification problem does not exist.

Therefore, when we restrict configurations to $\{0,1\}$, the all-conserving rule f must satisfy

$$\begin{aligned} &\forall x_1, x_2, x_3 \in \{0,1\}\colon\; f(x_1,x_2,x_3) = f_{184}(x_1,x_2,x_3)\,, \text{ or} \\ &\forall x_1, x_2, x_3 \in \{0,1\}\colon\; f(x_1,x_2,x_3) = f_{226}(x_1,x_2,x_3)\,. \end{aligned} \tag{8}$$

This is equivalent to saying that

$$f|_{01} = f_{184} \quad \text{or} \quad f|_{01} = f_{226}\,. \tag{9}$$

TABLE I

List of the Wolfram codes of the 15 all-conserving ternary rules (first column). Wolfram numbers of their binary projections are shown in columns $f|_{01}$, $f|_{02}$, and $f|_{12}$.

Wolfram number f	$f\|_{01}$	$f\|_{02}$	$f\|_{12}$
6213370633533	204	184	184
6768185473053	204	204	184
6924717700245	204	184	204
7479532539765	204	204	204
7486506443925	204	226	204
7573493966013	204	204	226
7580467870173	204	226	226
6914257071453	184	184	204
7469071910973	184	204	204
7563033337221	184	204	226
6769347793221	226	204	184
7480694859933	226	204	204
7487668764093	226	226	204
7625403764901	240	240	240
6159136430181	170	170	170

Exactly the same reasoning applies in the case of reductions to $\{0,1\}$, yielding the requirement

$$f|_{02} = f_{184} \text{ or } f|_{02} = f_{226} \,. \tag{10}$$

For configurations restricted to $\{1,2\}$, we obtain

$$f|_{12} = f_{184} \text{ or } f|_{12} = f_{226} \,. \tag{11}$$

A quick glance at Table I convinces us that there is no all-conserving rule satisfying simultaneously conditions of Eq. (9), (10), and (11). This demonstrates that an all-conserving f with the desired properties does not exist.

For absolute majority DCP, the proof is identical, as absolute majority and simple majority are the same when we restrict configurations to two states only. □

6. Interval-wise density classification

In addition to the variants of the DCP described in previous section, there exist yet another possible way to generalize the density classification problem which includes the "classical" binary DCP as a special case.

Suppose we want to classify finite strings of length L over the alphabet of M symbols $\mathcal{A} = \{0, 1, \dots, M-1\}$. Let

$$\rho(\boldsymbol{x}) = \frac{\sum_{i=0}^{L-1} x_i}{\sum_{i=0}^{L-1} \max \mathcal{A}}$$

be called *density* of configuration $\boldsymbol{x} = (x_0, x_2, \dots, x_{L-1})$, where $\max \mathcal{A}$ is the largest element of the alphabet $\mathcal{A}$, so that $\max \mathcal{A} = M-1$. This definition guarantees that $\rho \in [0,1]$, and obviously

$$\rho(\boldsymbol{x}) = \frac{1}{L(M-1)} \sum_{i=0}^{L-1} x_i \,.$$

Let $p_1, p_2, \dots, p_{M-1}$ be real numbers satisfying $0 < p_1 < p_2 < \dots < p_{M-1} < 1$. The pair of rules with global functions F and G solves interval-wise density classification problem if there exist integers n, m (possibly depending on L) such that for every configuration $\boldsymbol{x} = (x_0, x_2, \dots, x_{L-1})$, we have

$$\begin{aligned}
G^m F^n(\boldsymbol{x}) &= 0^L \text{ if } \rho(\boldsymbol{x}) \in [0, p_1) \,, \\
G^m F^n(\boldsymbol{x}) &= 1^L \text{ if } \rho(\boldsymbol{x}) \in (p_1, p_2) \,, \\
G^m F^n(\boldsymbol{x}) &= 2^L \text{ if } \rho(\boldsymbol{x}) \in (p_2, p_3) \,, \\
&\dots \\
G^m F^n(\boldsymbol{x}) &= (M-1)^L \text{ if } \rho(\boldsymbol{x}) \in (p_{M-1}, 1] \,.
\end{aligned}$$

Obviously, when $M = 2$ and $p_1 = 1/2$, the above reduces to the "classical" binary density classification problem, with the known solution by the pair of ECA 184 and 232. Note that we intentionally made the intervals open at points $p_1, p_2, \ldots, p_{M-1}$, to be compatible with the standard DCP, where the classification of strings with equal number of zeros and ones is not required.

When $p_i = 1/M$ for $i = 1, 2, \ldots, M-1$, we call the problem *symmetric interval-wise* density classification problem. For ternary rules ($M = 3$), does a two-rule solution of the symmetric interval-wise DCP exist? Again, based on our extensive heuristic search, we suspect that the answer is no, but we were able to prove the non-existence only for rules reducible to two states.

Suppose that such a solution indeed existed, consisting of rules with global functions F and G, both reducible to two states. This would mean that for some m and n, we have

$$\begin{aligned}
G^m F^n(\boldsymbol{x}) &= 0^L \text{ if } \rho(\boldsymbol{x}) \in [0, 1/3)\,,\\
G^m F^n(\boldsymbol{x}) &= 1^L \text{ if } \rho(\boldsymbol{x}) \in (1/3, 2/3)\,,\\
G^m F^n(\boldsymbol{x}) &= 2^L \text{ if } \rho(\boldsymbol{x}) \in (2/3, 1]\,.
\end{aligned}$$

Now, let us suppose that $\boldsymbol{x}$ is binary, consisting only of 0s and 1s. In such a case, the above would reduce to

$$\begin{aligned}
G^m F^n(\boldsymbol{x}) &= 0^L \text{ if } 0 \le \rho(\boldsymbol{x}) < 1/3\,,\\
G^m F^n(\boldsymbol{x}) &= 1^L \text{ if } 1/3 < \rho(\boldsymbol{x}) < 2/3\,,
\end{aligned}$$

or, using the definition of $\rho(\boldsymbol{x}) = \frac{1}{2L} \sum x_i$,

$$\begin{aligned}
G^m F^n(\boldsymbol{x}) &= 0^L \text{ if } 0 \le \frac{1}{L} \sum_{i=0}^{L-1} x_i < 2/3\,,\\
G^m F^n(\boldsymbol{x}) &= 1^L \text{ if } 1/3 < \frac{1}{L} \sum_{i=0}^{L-1} x_i < 4/3\,.
\end{aligned}$$

This would imply existence of a pair of elementary rules having the property that $G^m F^n(\boldsymbol{x})$ consists of all zeros if zeros occupy 2/3 of all sites of the initial configuration and all ones otherwise. No such pair of elementary rules exists, and it can be verified numerically by checking all possible cases, as there are only 256 elementary rules, yielding $256 \times 256 = 65\,536$ possible pairs. This proves the following.

Proposition 4 *There is no pair of nearest-neighbour ternary rules reducible to two states which would solve the symmetric interval-wise density classification problem.*

7. Non-symmetric interval-wise classification

If there is no solution of the symmetric interval-wise problem, can we at least perform non-symmetric classification with two ternary rules?

We performed an intensive heuristic search for such rules and after some tinkering with rule tables, by trial and error, we found the following interesting pair of ternary rules.

Conjecture 1 *Let F be the ternary nearest-neighbour rule with Wolfram number 6478767664173, and G be the rule with Wolfram number 7580606234490. For any finite ternary string $\boldsymbol{x}$ of length L, containing at least one zero, let* $\rho = \frac{1}{2L}\sum_{i=0}^{L-1} x_i$. *Then*

$$\begin{aligned} G^L F^L(\boldsymbol{x}) &= 0^L \text{ if } \rho(\boldsymbol{x}) \in [0, 2/3)\,, \\ G^L F^L(\boldsymbol{x}) &= 1^L \text{ if } \rho(\boldsymbol{x}) \in (2/3, 3/4)\,, \\ G^L F^L(\boldsymbol{x}) &= 2^L \text{ if } \rho(\boldsymbol{x}) \in (3/4, 1)\,. \end{aligned}$$

This means that the pair of rules (6478767664173, 7580606234490) "almost" solves the interval-wise DCP with $p_1 = 2/3$ and $p_2 = 3/4$. We use the word "almost" because it only works for configurations which contain at least one zero. Configurations which do not satisfy this property can be misclassified — for example, 1^L has density 1/2, thus should produce 0^L in the end, yet 1^L is a fixed point of both rules 6478767664173 and 7580606234490[1].

We performed extensive numerical experiments to verify the above conjecture, and it appears to be valid. Below, we provide a sketch of a possible proof of this result. Figures 1 and 2 show definitions of rules F and G. The first of them (rule 6478767664173) is number-conserving, and its binary projection $f|_{12}$ (defined as in Eq. (7)) is ECA 184. This rule plays an analogous role as rule 184 in the two-rule solution of the binary DCP, namely, it prepares the configuration for further processing without changing its density $\rho(\boldsymbol{x})$. After sufficiently many iterations (which we simply take to be L), this rule eliminates certain symbols and substrings, as shown in Table II.

One can see that, for example, when $\rho(\boldsymbol{x}) < 2/3$, after L iterations of rule F, the configuration may contain substrings 00 and 11 as well as symbols 0, 1, and 2, while substrings 22 are always absent. When $\rho(\boldsymbol{x}) \in (2/3, 3/4)$, the configuration may contain substrings 11 and symbols 1 and 2, while substrings 00 and 22 as well as 0s are absent. And, finally, when $\rho(\boldsymbol{x}) > 3/4$, substrings 22 may be present, 00 and 11 are absent, while symbols 1 and 2 may be present and 0s are absent.

[1] We wish to thank anonymous referee for pointing out this fact.

$f(0,0,0)=0$	$f(0,0,0)=0$	$f(1,1,1)=\mathbf{1}$	
$f(0,0,1)=0$	$f(0,0,2)=1$	$f(1,1,2)=\mathbf{1}$	
$f(0,1,0)=1$	$f(0,2,0)=1$	$f(1,2,1)=\mathbf{1}$	$f(0,1,2)=2$
$f(0,1,1)=2$	$f(0,2,2)=2$	$f(1,2,2)=\mathbf{2}$	$f(0,2,1)=1$
$f(1,0,0)=0$	$f(2,0,0)=0$	$f(2,1,1)=\mathbf{2}$	$f(1,0,2)=1$
$f(1,0,1)=0$	$f(2,0,2)=1$	$f(2,1,2)=\mathbf{2}$	$f(1,2,0)=1$
$f(1,1,0)=0$	$f(2,2,0)=1$	$f(2,2,1)=\mathbf{1}$	$f(2,0,1)=0$
$f(1,1,1)=1$	$f(2,2,2)=2$	$f(2,2,2)=\mathbf{2}$	$f(2,1,0)=1$

Fig. 1. Definition of rule 6478767664173. Entries with bold outputs correspond to restriction to configurations consisting only of 1s and 2s, on which the above rule is equivalent to rule 184.

$f(0,0,0)=0$	$f(0,0,0)=0$	$f(1,1,1)=1$	
$f(0,0,1)=0$	$f(0,0,2)=0$	$f(1,1,2)=1$	
$f(0,1,0)=0$	$f(0,2,0)=0$	$f(1,2,1)=1$	$f(0,1,2)=0$
$f(0,1,1)=0$	$f(0,2,2)=2$	$f(1,2,2)=2$	$f(0,2,1)=0$
$f(1,0,0)=0$	$f(2,0,0)=0$	$f(2,1,1)=1$	$f(1,0,2)=0$
$f(1,0,1)=0$	$f(2,0,2)=2$	$f(2,1,2)=2$	$f(1,2,0)=2$
$f(1,1,0)=1$	$f(2,2,0)=2$	$f(2,2,1)=2$	$f(2,0,1)=1$
$f(1,1,1)=1$	$f(2,2,2)=2$	$f(2,2,2)=2$	$f(2,1,0)=1$

Fig. 2. Definition of rule 7580606234490. Its binary projections are ECA 192, 232, and 232, and this can be verified by inspecting the first three columns.

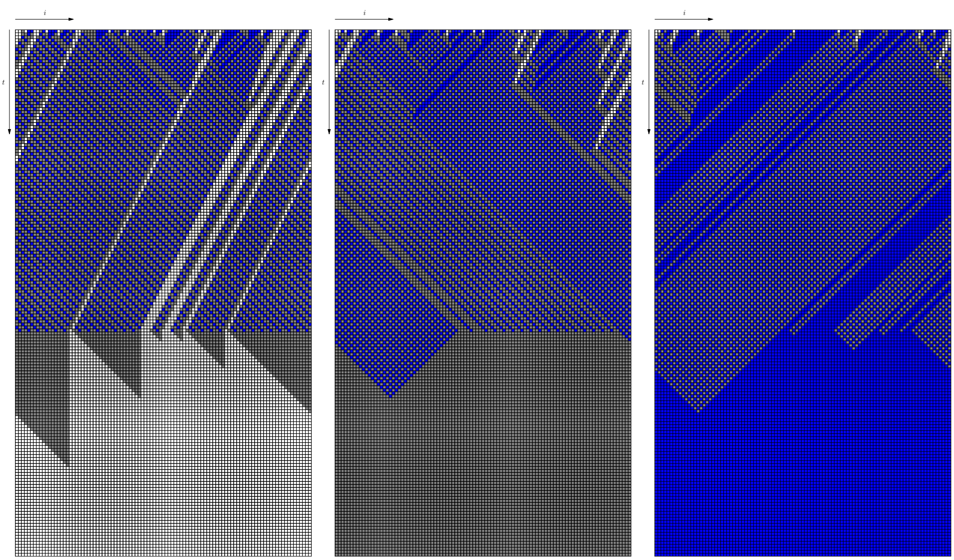
TABLE II

Presence of selected substrings in the final configuration after iterating rule 6478767664173 L times.

Substring	$\rho(\boldsymbol{x}) < 2/3$	$\rho(\boldsymbol{x}) \in (2/3, 3/4)$	$\rho(\boldsymbol{x}) > 3/4$
00	Yes	No	No
11	Yes	Yes	No
22	No	No	Yes
0	Yes	No	No
1	Yes	Yes	Yes
2	Yes	Yes	Yes

The second rule G, with Wolfram number 6478767664173, plays a role similar to rule 232 in the two-rule solution of the binary DCP. Its three binary projections $f|_{01}$, $f|_{02}$ and $f|_{12}$ exist, and their Wolfram numbers are, respectively, 192, 232, and 232. This rule grows clusters of 0s if they are present, and if not, it just behaves like rule 232, that is, grows clusters of 1s in the absence of pairs 22, and grows clusters of 2s in the absence of pairs 11. A quick look at Table II reveals that these three cases will occur after we iterate rule F on initial configurations with densities, respectively, $\rho(\boldsymbol{x}) < 2/3$, $\rho(\boldsymbol{x}) \in (2/3, 3/4)$, and $\rho(\boldsymbol{x}) > 3/4$. Note that presence of 0s is required to grow clusters of 0s, thus the need for additional condition imposed in the initial configuration (it must contain at least one zero).

The final effect, therefore, of iterations of rule G starting with $F^L(\boldsymbol{x})$ will be all zeros when $\rho(\boldsymbol{x}) < 2/3$, all ones when $\rho(\boldsymbol{x}) \in (2/3, 3/4)$, and all 2s when $\rho(\boldsymbol{x}) > 3/4$, exactly as claimed in Conjecture 1. Examples of three cases of density classification by the aforementioned rules are shown in Fig. 3.



$\rho(\boldsymbol{x}) = 0.6$ $\rho(\boldsymbol{x}) = 0.7$ $\rho(\boldsymbol{x}) = 0.8$

Fig. 3. (Colour on-line) Spatiotemporal patterns illustrating a two-rule solution of non-symmetric interval-wise DCP by rules 6478767664173 and 7580606234490. White color represents 0s, grey 1s, and dark grey/blue 2s.

Obviously, the above is only a sketch of a proof and it needs further elaboration. Our statement about rules 6478767664173 and 7580606234490, therefore, must remain a conjecture for now.

8. Conclusions and future work

We have demonstrated that except the trivial case of 0-majority, there exists no two-rule solution of various density classification problems (simple majority, absolute majority, symmetric interval-wise) in the domain of ternary nearest-neighbour rules which would be analogous to the known solution of DCP by the pair of ECA 184 and 232. By "analogous" we mean a solution consisting of two rules reducible to two states in which the first rule serves as a pre-processor preserving relationship between densities, thus is all-conserving for simple/absolute majority problem, or number-conserving for interval-wise problem.

This naturally brings up a question if two-rule solutions exist if one relaxes the restriction of the first rule possessing additive invariant or the restriction of rules being reducible. While such a possibility cannot be excluded, we seriously doubt it. However, it seems quite possible that extending the neighbourhood size to two nearest neighbours may help to produce a two-rule solution. We plan to investigate this possibility in the near future.

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