MAJORITY-VOTE MODEL WITH INDEPENDENT AGENTS ON COMPLEX NETWORKS*

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Majority-vote model with independence is investigated on complex scale-free networks with degree distribution $p_k \propto k^{\tilde{\gamma}}$. In the simplest version of the majority-vote model, the agents assume, with probability $1 - q$ ($0 < q < 1/2$), the opinion in agreement with that of the majority of their neighbors. In the majority-vote model with independence, the agents obey the above-mentioned update rule with probability $1 - \tilde{p}$ ($0 < \tilde{p} < 1$), while with probability $\tilde{p}$, they make decision randomly. It is shown that the parameters q and $p = \tilde{p}/2$ are equivalent, and as one of them is decreased, with the other fixed, for $\tilde{\gamma} > 5/2$, the model can exhibit transition to the ferromagnetic state at a critical value $q_{\rm c}$ or $p_{\rm c}$ which depends on the degree distribution. Critical behavior of the magnetization is determined in the heterogeneous mean-field approximation. For $5/2 < \tilde{\gamma} < 7/2$, this behavior is non-universal, with the magnetization scaling as $M \propto (q_{\rm c} - q)^{\beta}$ or $M \propto (p_{\rm c} - p)^{\beta}$ with $\beta = 1/[2(\tilde{\gamma} - 5/2)]$, and for $\tilde{\gamma} \geq 7/2$, it becomes universal with $\beta = 1/2$. These results are confirmed by Monte Carlo simulations and finite size scaling analysis.

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1. Introduction

The majority-vote (MV) model [1, 2] is a stochastic model for the opinion formation in which agents represented by two-state spins update their opinions at discrete time steps, with certain probability following the opinion of the majority of their neighbors or, otherwise, following the opposite opinion. In typical variants of the MV model, the only parameter is q, $0 \leq q \leq 1/2$, which determines the above-mentioned probability $1 - q$ to obey the majority rule and, thus, controls the level of internal noise. The MV model is a nonequilibrium counterpart of the Ising model since the update rules for the spins depend only on the state of their neighbors (there

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is no global Hamiltonian), and the model does not obey the detailed balance condition. Nevertheless, it was shown that the MV model on regular d-dimensional lattices, $d = 2, 3 \ldots$, [1, 3–5] as well as on different complex networks [6–13] exhibits a second-order phase transition from a paramagnetic (PM) state to a ferromagnetic (FM) state at finite q_c. Recently, the MV model has been extended by introducing another kind of internal noise, related to the independence of agents' behavior [14] and controlled by another parameter $\tilde{p}$ $(0 < \tilde{p} < 1)$. In this case, the agents updating their opinions with probability $1 - \tilde{p}$ follow the above-mentioned typical majority rule and with probability $\tilde{p}$ make decision randomly. In this variant of the MV model, the FM transition can occur either at $q = q_c$ for fixed $\tilde{p}$ or at $\tilde{p} = \tilde{p}_c$ for fixed q. Other extensions of the MV model which also enrich its critical behavior comprise, *e.g.*, inclusion of heterogeneous agents [15], agents with more than two opinions [16], agents with inertia (which leads to the occurrence of a discontinuous FM transition) [17], and anticonformist agents (which leads to antiferromagnetic or spin-glass-like rather than FM transition) [18, 19].

Investigation of the MV model has been focused primarily on its critical properties in the vicinity of the FM transition point. This is mainly because the hypothesis that non-equilibrium stochastic spin systems with up–down symmetry on regular lattices can exhibit second-order phase transition and belong to the universality class of the corresponding equilibrium Ising model [20]. Indeed, the MV model on two- and three-dimensional regular lattices was shown to fulfil this hypothesis [1, 3–5]; the same concerns the MV model with independent agents on two-dimensional lattices [14]. Besides, it was also demonstrated that an upper critical dimension exists above which the MV model on regular lattices belongs to the Ising mean-field (MF) universality class [4]. In contrast, the critical properties of the MV model on complex networks, such as Erdös–Rényi graphs [21–23] or heterogeneous scale-free (SF) networks [22, 23], can differ from those of the corresponding Ising model even in the case of networks with high mean degree of nodes (mean number of edges per node) [6, 7, 11]. In particular, it was found that the MV model on SF networks exhibits FM phase transition at finite q_c for a wider class of such networks than the Ising model, *e.g.*, in the case of networks with diverging second moment of the degree distribution (distribution of the degrees of nodes) [11] in which the critical temperature for the Ising model diverges [24–28]. In order to study the critical properties of the MV model on heterogeneous SF networks in more detail, in this paper, the model with independent agents is investigated. The critical values of the parameters q_c, $\tilde{p}_c$ for the FM transition as well as the critical exponent for the magnetization are evaluated in the framework of the heterogeneous MF theory. In particular, it is shown that the critical behavior of the model below the transition point as one of the parameters q, $\tilde{p}$ is varied does not depend on the other, fixed parameter, but can depend on the degree distribution. Analytic results are confirmed by Monte Carlo (MC) simulations and finite size scaling (FSS) analysis for the MV model on SF networks with high mean degree.

2. The model

The MV model consists of agents represented by two-state spins $s_i = \pm 1$, $i = 1, 2, \ldots N$ located in the nodes of a (possibly complex) network, updating their opinions (orientations) at discrete time steps with probability dependent on the opinion of the majority of their neighbors. In most cases considered so far, the agents obey the FM update rule in which the probability of the opinion flip of the spin s_i per unit time (rate) is

$$w_i(\boldsymbol{s}) = \frac{1}{2}\left[1 - (1-2q)s_i \mathrm{sign}_i\right] , \tag{1}$$

where $0 < q < 1/2$ is the parameter which controls the level of internal noise, $\boldsymbol{s}$ denotes the spin configuration of the model,

$$\mathrm{sign}_i = \mathrm{sign}\left(\sum_{j \in nn_i} s_j\right) , \qquad \mathrm{sign}(x) = \begin{cases} -1 & \text{for} \quad x < 0 , \\ 0 & \text{for} \quad x = 0 , \\ +1 & \text{for} \quad x > 0 , \end{cases} \tag{2}$$

and nn_i denotes a set of the nearest neighbors of the node i, *i.e.*, of the nodes which are directly connected by edges with the node i. According to this update rule, the agents follow the opinion of the majority of their neighbors with probability $1 - q$ and the opposite opinion with probability q. Hence, the parameter q is a measure of the agents' uncertainity in decision making.

In Ref. [14], a modification of the above-mentioned model was proposed in which the agents follow the FM update rule (1) with probability $1 - \tilde{p}$, while with probability $\tilde{p}$, they act independently and make decision randomly, choosing one of the two possible orientations with equal probability. The modified spin-flip rate is then

$$w_i(\boldsymbol{s}) = \frac{1-\tilde{p}}{2}\left[1 - (1-2q)s_i \mathrm{sign}_i\right] + \frac{\tilde{p}}{2} . \tag{3}$$

Let us note that introducing parameter $p = \tilde{p}/2$ this rate can be written in a more convenient form

$$w_i(\boldsymbol{s}) = \frac{1}{2}\left[1 - (1-2p)(1-2q)s_i \mathrm{sign}_i\right] , \tag{4}$$

which is symmetric with respect to the exchange of the parameters p, q. This means that the critical behavior of the model (occurrence of the FM transition, scaling of the order parameter, *etc.*) with fixed p and decreasing q and *vice versa* is identical.

So far, the MV model with the spin-flip rate given by Eq. (4) has been studied on complete graphs and 2-dimensional regular lattices [14]. In this paper, it is investigated on complex networks characterized by the degree distribution p_k with the mean degree $\langle k \rangle$. In particular, the case of heterogeneous SF networks is considered [22, 23] in which

$$p_k = Ak^{-\tilde{\gamma}} \quad \text{for} \quad k \geq m \,, \qquad A = (\tilde{\gamma} - 1)\, m^{\tilde{\gamma}-1} \,, \tag{5}$$

where $\tilde{\gamma} > 2$ and m is the minimum degree of nodes. As can be deduced from Eq. (4), the MV model under study is equivalent to the kinetic Ising model with the Glauber spin-flip rate attached to two thermal baths, the one with zero temperature with probability $(1-2p)(1-2q)$ and the other one with infinite temperature with probability $1-(1-2p)(1-2q)$; thus, it is a non-equilibrium counterpart of the Ising model with non-zero heat flux. Hence, it seems interesting to compare the critical properties of the Ising and the MV models on SF networks. It is known that the Ising model on SF networks shows FM transition for $\tilde{\gamma} > 3$ and non-universal critical behavior for $3 < \tilde{\gamma} < 5$ [24–28], while for $\tilde{\gamma} > 5$, it belongs to the standard MF universality class. Below, it is demonstrated that the MV model with independent agents on SF networks shows FM transition for $\tilde{\gamma} > 5/2$ and non-universal critical behavior, different than the Ising model, for $5/2 < \tilde{\gamma} < 7/2$, while for $\tilde{\gamma} > 7/2$, it belongs to the standard MF universality class.

3. Heterogeneous mean-field approximation

In this section, the critical values of the parameters q_{c} and p_{c} are evaluated for the FM transition in the MV model with independence on SF networks with fixed p and q, respectively. This is done in the framework of the heterogeneous MF approximation for which the natural order parameter is $S = \sum_{i=1}^{N} k_i \langle s_i \rangle$ rather than usual magnetization, where $\langle s_i \rangle$ is the mean value of the spin in node i and k_i is the degree of the node i. Using this approximation, the critical value q_{c} was obtained for the MV model with $p = 0$ (*i.e.*, with agents without independence) on complex heterogeneous networks [11]. Here, the calculations are repeated to include also the case with independent agents with $p > 0$.

For the spin system with dynamics governed by the transition rates $w\left(\boldsymbol{s}|\, \boldsymbol{s}'\right)$ from the spin configuration $\boldsymbol{s}'$ to $\boldsymbol{s}$, the Master equation for the probability $P(\boldsymbol{s}, t)$ that at time t the spin configuration is $\boldsymbol{s}$ has a general form of

$$\frac{\mathrm{d}P\left(\boldsymbol{s}, t\right)}{\mathrm{d}t} = \sum_{\boldsymbol{s}'} \left[w\left(\boldsymbol{s}|\, \boldsymbol{s}'\right) P\left(\boldsymbol{s}', t\right) - w\left(\boldsymbol{s}'|\, \boldsymbol{s}\right) P\left(\boldsymbol{s}, t\right) \right] . \tag{6}$$

Taking into account that at each time step, transition occurs between spin configurations $\boldsymbol{s}' \to \boldsymbol{s}$ differing just by one spin flipped, say s_i, thus the transition rate $w\left(\boldsymbol{s}|\,\boldsymbol{s}'\right) = w_i\left(\boldsymbol{s}'\right)$ is given by Eq. (4), and performing ensemble average of the Master equation (6), the following equation is obtained

$$\frac{\partial\langle s_i\rangle}{\partial t} = -2\langle s_i w_i\left(\boldsymbol{s}\right)\rangle = -\langle s_i\rangle + (1-2p)(1-2q)\langle \mathrm{sign}_i\rangle\,, \tag{7}$$

with

$$\langle \mathrm{sign}_i\rangle = (+1)\,\mathrm{Pr}\left(\mathrm{sign}_i = +1\right) + (-1)\,\mathrm{Pr}\left(\mathrm{sign}_i = -1\right)\,. \tag{8}$$

Let us consider the MV model on a general heterogeneous network, where the nodes are characterized by their degrees k_i which are drawn from a given probability distribution p_k. Then

$$\mathrm{Pr}\left(\mathrm{sign}_i = \pm 1\right) = \sum_{l=\left[\frac{k_i}{2}\right]}^{k_i}\binom{k_i}{l}\prod_{j}^{l}\mathrm{Pr}\left(s_j = \pm 1\right)\prod_{j\prime}^{k_i-l}\mathrm{Pr}\left(s_{j\prime} = \mp 1\right)\,, \tag{9}$$

where $[\cdot]$ denotes the ceil function and j, j' denote nodes connected with the node i by edges.

In order to evaluate the probabilities in Eq. (9), heterogeneous MF approximation will be applied. The main assumption for this approximation is that the nodes can be divided into classes according to their degrees and that spins in the nodes belonging to the same class are equivalent. Then for the spin s_i located in the node with degree k_i, there is

$$\mathrm{Pr}\left(s_i = \pm 1\right) = \frac{1 \pm \langle s_i\rangle}{2} = \frac{1 \pm \langle s_{k_i}\rangle}{2}\,, \tag{10}$$

where $\langle s_{k_i}\rangle$ denotes the mean value of the spin in each node with degree k_i. Progress in analytic considerations can be achieved in the case of uncorrelated networks, *i.e.*, networks without correlations between degrees of nodes. Then the probability that an edge attached to node i points at the other end at the node with degree k is $kp_k/\langle k\rangle$. Thus, among l nodes in the first product in Eq. (9), containing spins with the same orientation as that of the spin in node i, formally there are $lkp_k/\langle k\rangle$ nodes belonging to a given class of nodes with degree k; similarly, among $k_i - l$ nodes in the second product in Eq. (9), containing spins with opposite orientation than that of the spin in node i, formally there are $(k_i - l)\,kp_k/\langle k\rangle$ nodes belonging to a class of nodes with degree k. As a result, products over the indices of nodes j, j' in Eq. (9) can be replaced with the product over the degrees (classes) of nodes k. Namely, using Eq. (10), the probabilities in Eq. (9) can be written as

$$\mathrm{Pr}\left(\mathrm{sign}_i = \pm 1\right) = \sum_{l=\left[\frac{k_i}{2}\right]}^{k_i}\binom{k_i}{l}\prod_{k}\left(\frac{1 \pm \langle s_k\rangle}{2}\right)^{l\frac{kp_k}{\langle k\rangle}}\left(\frac{1 \mp \langle s_k\rangle}{2}\right)^{(k_i-l)\frac{kp_k}{\langle k\rangle}}\,. \tag{11}$$

Approximating $(1 \pm \langle s_k \rangle)^{\frac{kp_k}{\langle k \rangle}} \approx 1 \pm \frac{kp_k}{\langle k \rangle} \langle s_k \rangle$ and introducing the order parameter

$$S = (N\langle k \rangle)^{-1} \sum_{i=1}^{N} k_i \langle s_i \rangle = \sum_k \frac{kp_k}{\langle k \rangle} \langle s_k \rangle \,, \tag{12}$$

where, again, the summation over the indices of nodes in the first sum of Eq. (12) is replaced with the sum over the degrees (classes) of nodes k, it is obtained that

$$\Pr\left(\mathrm{sign}_i = \pm 1\right) = \sum_{l=\left[\frac{k_i}{2}\right]}^{k_i} \binom{k_i}{l} \left(\frac{1 \pm S}{2}\right)^{l} \left(\frac{1 \mp S}{2}\right)^{k_i - l} . \tag{13}$$

For large k_i, the binomial distribution in Eq. (13) can be approximated by the normal distribution, thus

$$\Pr\left(\mathrm{sign}_i = \pm 1\right) \approx \frac{1}{2} \pm \frac{1}{2} \mathrm{erf}\left(\frac{S}{2}\sqrt{2k_i}\right) . \tag{14}$$

Substituting Eq. (14) into Eq. (8) and then into Eq. (7), it is obtained that for spins located in nodes with large degrees

$$\frac{\partial \langle s_i \rangle}{\partial t} = -\langle s_i \rangle + (1-2p)(1-2q)\mathrm{erf}\left(\sqrt{\frac{k_i}{2}} S\right) . \tag{15}$$

Multiplying both sides of Eq. (15) by k_i, performing summation over all nodes and taking into account Eq. (12), the following equation for the time dependence of the order parameter is obtained:

$$\frac{\partial S}{\partial t} = -S + (1-2p)(1-2q) \sum_k \frac{kp_k}{\langle k \rangle} \mathrm{erf}\left(\sqrt{\frac{k}{2}} S\right) . \tag{16}$$

The fixed points of Eq. (16) are solutions of the equation

$$S = (1-2p)(1-2q) \sum_k \frac{kp_k}{\langle k \rangle} \mathrm{erf}\left(\sqrt{\frac{k}{2}} S\right) . \tag{17}$$

For any p, q, Eq. (17) has a solution $S_0 = 0$ corresponding to the PM phase. Stability of this solution can be investigated by assuming $S = S_0 + \xi = \xi$, where ξ is a small deviation from the fixed point S_0 and linearizing the right-hand side of Eq. (16) with respect to ξ which yields

$$\frac{\partial \xi}{\partial t} = -\left[1 - (1-2q)(1-2p)\frac{2}{\sqrt{2\pi}} \frac{\langle k^{3/2} \rangle}{\langle k \rangle}\right] \xi \,, \tag{18}$$

where $\langle k^{3/2}\rangle = \sum_k k^{3/2} p_k$ is the moment of the order of 3/2 of the degree distribution p_k. Thus, for fixed p and for decreasing q, the fixed point $S_0 = 0$ loses stability and the transition from the PM to the FM phase occurs at

$$q_{\mathrm{c}} = \frac{1}{2}\left[1 - \frac{\sqrt{2\pi}}{2(1-2p)}\frac{\langle k\rangle}{\langle k^{3/2}\rangle}\right] \tag{19}$$

provided that $q_{\mathrm{c}} > 0$ in Eq. (19) which is true for small enough p; otherwise, there is no transition and the model remains in the PM state for the whole range $0 < q < 1/2$. Similarly, for fixed q and for decreasing p, the fixed point $S_0 = 0$ loses stability and the transition from the PM to the FM phase occurs at

$$p_{\mathrm{c}} = \frac{1}{2}\left[1 - \frac{\sqrt{2\pi}}{2(1-2q)}\frac{\langle k\rangle}{\langle k^{3/2}\rangle}\right] \tag{20}$$

provided that $p_{\mathrm{c}} > 0$ in Eq. (20) which is true for small enough q; otherwise, there is no transition and the model remains in the PM state for the whole range $0 < p < 1/2$. It can be seen that Eqs. (19) and (20) are symmetric with respect to exchanging p and q, as expected.

Expressions (19), (20) for the critical values q_{c}, p_{c} are valid for the MV model under study on any network with large enough mean degree of nodes. In particular, in the case of SF networks with the degree distribution (5), the moment $\langle k^{3/2}\rangle$ diverges for $\tilde{\gamma} \le 5/2$, thus the FM transition can occur at $0 < q_{\mathrm{c}} < 1/2$ or $0 < p_{\mathrm{c}} < 1/2$ only in the model on networks with $\tilde{\gamma} > 5/2$ for which

$$\langle k\rangle = \frac{\tilde{\gamma}-1}{\tilde{\gamma}-2}m\,, \qquad \left\langle k^{3/2}\right\rangle = \frac{\tilde{\gamma}-1}{\tilde{\gamma}-5/2}m^{3/2}\,. \tag{21}$$

4. Critical behavior of the order parameter and the magnetization

The critical properties of the MV model on heterogeneous SF networks so far have not been investigated in detail. In Ref. [11], critical behavior of the susceptibility was studied semi-analytically in the heterogeneous MF approximation, and by MC simulations and FSS analysis. However, in order to determine the universality class of the model under study, the critical behavior of the order parameter below the FM transition point should be also determined. In the case of the FM transition with fixed p, the expected scaling behavior of the magnetization M is $M \propto \varepsilon_q^{\beta}$, where $\varepsilon_q = (q_{\mathrm{c}} - q)/q_{\mathrm{c}}$; similarly, in the case of the FM transition with fixed q, it is $M \propto \varepsilon_p^{\beta}$, where $\varepsilon_p = (p_{\mathrm{c}} - p)/p_{\mathrm{c}}$, with the same critical exponent β (possibly dependent on the degree distribution) due to the symmetry of the spin-flip rate, Eq. (4), with respect to the exchange of the parameters p and q. Below, the critical exponent β is evaluated in the heterogeneous MF approximation for various degree distributions of the underlying SF network of interactions.

4.1. Networks with finite moments of the degree distribution

For the sake of simplicity, let us first consider the MV model on networks with all moments of the degree distribution finite; this case comprises, *e.g.*, regular d-dimensional lattices with $d \geq 2$, Erdös–Rényi graphs [21–23], *etc.* Then the scaling S *vs.* ε_q can be obtained by expanding the right-hand side of Eq. (17) in the Taylor series. Denoting $f(u) = \mathrm{erf}(u) = \sum_{n=0}^{\infty} f_n u^n$, where $f_n = \frac{1}{n!} \left.\frac{\mathrm{d}^n f}{\mathrm{d}u^n}\right|_0$, this yields

$$S = (1-2p)(1-2q) \sum_{n=0}^{\infty} \frac{f_n}{2^{n/2}} \frac{\langle k^{n/2+1} \rangle}{\langle k \rangle} S^n \,, \tag{22}$$

where $\langle k^{n/2+1} \rangle$ are moments of the order of $n/2+1$ of the degree distribution p_k. Taking into account that $f_0 = f_2 = 0$, $f_1 = 2/\sqrt{\pi}$, $f_3 = -4/\left(3!\sqrt{\pi}\,\right)$, the scaling S *vs.* ε_q for fixed p can be obtained from Eq. (22) by taking the limit $q \to q_{\mathrm{c}}$ ($\varepsilon_q \to 0$) and retaining on the right-hand side only the linear and the dominant nonlinear term of the order of S^3. This yields

$$-\frac{4q_{\mathrm{c}}}{\sqrt{2\pi}} \frac{\langle k^{3/2} \rangle}{\langle k \rangle} \varepsilon_q = -\frac{(1-2q_{\mathrm{c}})}{3\sqrt{2\pi}} \frac{\langle k^{5/2} \rangle}{\langle k \rangle} S^2 + \ldots \tag{23}$$

which results in the typical MF scaling $S \propto \varepsilon_q^{1/2}$. The scaling for the magnetization $M = N^{-1} \sum_{i=1}^{N} \langle s_i \rangle$ is the same, $M \propto \varepsilon_q^{1/2}$, since in this case, $M \propto S$. Since the factors $1-2p$ cancel on both sides of Eq. (23), the expected scaling for S and M *vs.* ε_q does not depend on p. Besides, due to the symmetry of Eq. (22) with respect to the exchange of q and p, it is obvious that for fixed q, the scaling of S and M with respect to ε_p is the same as that with respect to ε_q for fixed p. The expected value of the critical exponent $\beta = 1/2$ agrees well with that obtained from MC simulations of the MV model on random Erdös–Rényi graphs with high mean degree [6].

4.2. Heterogeneous scale-free networks

In the case of the MV model on heterogeneous SF networks, the scaling S *vs.* ε_q cannot be obtained by simply expanding the right-hand side of Eq. (17) in the Taylor series with respect to small parameter S. This is because higher-order moments $\langle k^{n/2+1} \rangle$ present in Eq. (22) diverge and compete with small terms S^n making the convergence of the series problematic.

Nevertheless, the right-hand side of Eq. (17) also in this case can be written in a form of a converging power series using a more elaborate method of Ref. [26].

Replacing summation by integration and substituting $u = S\sqrt{k/2}$ with p_k as in Eq. (5), Eq. (17) can be written as

$$\begin{aligned} S &= (1-2p)\,(1-2q)\,\frac{A}{\langle k\rangle}\int\limits_{m}^{\infty} k^{-\tilde{\gamma}+1}\mathrm{erf}\left(\sqrt{\frac{k}{2}}S\right)\mathrm{d}k \\ &= (1-2p)\,(1-2q)\,\frac{A}{\langle k\rangle}\,\frac{2^{-\tilde{\gamma}+3}}{S^{-2\tilde{\gamma}+4}}\int\limits_{\frac{S}{\sqrt{2}}m^{1/2}}^{\infty}\frac{\mathrm{erf}(u)}{u^{2\tilde{\gamma}-3}}\mathrm{d}u\,. \end{aligned} \tag{24}$$

Let m_0 denote an integer number such that $m_0 < 2\tilde{\gamma}-3 < m_0+1$; since phase transition occurs for $\tilde{\gamma} > 5/2$, there is $m_0 \geq 3$. Denoting again $f(u) = \mathrm{erf}(u)$, *etc.*, the integral on the right-hand side of Eq. (24) can be evaluated by expanding $f(u)$ in the Taylor series

$$f(u) = \sum_{n=0}^{\infty} f_n u^n = \sum_{n=0}^{m_0-1} f_n u^n + \hat{f}(u)\,, \tag{25}$$

where $\hat{f}(u) = f(u) - \sum_{n=0}^{m_0-1} f_n u^n = \sum_{n=m_0}^{\infty} f_n u^n$. This yields

$$\begin{aligned} \int\limits_{\frac{S}{\sqrt{2}}m^{1/2}}^{\infty}\frac{\mathrm{erf}(u)}{u^{2\tilde{\gamma}-3}}\mathrm{d}u &= \sum_{n=0}^{m_0-1} f_n \int\limits_{\frac{S}{\sqrt{2}}m^{1/2}}^{\infty} u^{n-2\tilde{\gamma}+3}\mathrm{d}u \\ &\quad + I(\tilde{\gamma}) + \sum_{n=m_0}^{\infty} f_n \int\limits_{0}^{\frac{S}{\sqrt{2}}m^{1/2}} u^{n-2\tilde{\gamma}+3}\mathrm{d}u\,, \end{aligned} \tag{26}$$

where $I(\tilde{\gamma}) = \int_0^{\infty}\frac{\hat{f}(u)}{u^{2\tilde{\gamma}-3}}\mathrm{d}u < 0$ converges since $\frac{\hat{f}(u)}{u^{2\tilde{\gamma}-3}} \propto u^{m_0-(2\tilde{\gamma}+3)}$ for $u \to 0$ and $\frac{\hat{f}(u)}{u^{2\tilde{\gamma}-3}} \propto u^{-(2\tilde{\gamma}+3)}$ for $u \to \infty$, and the remaining integrals are trivial. Hence, the right-hand side of Eq. (24) can be written in a form of a converging series, so that

$$\begin{aligned} S &= 2^{-\tilde{\gamma}+3}(1-2p)\,(1-2q)\,\frac{A}{\langle k\rangle}I(\tilde{\gamma})S^{2\tilde{\gamma}-4} \\ &\quad -2m^{2-\tilde{\gamma}}(1-2p)\,(1-2q)\,\frac{A}{\langle k\rangle}\sum_{n=0}^{\infty}\frac{(m/2)^{n/2}f_n}{n+1-(2\tilde{\gamma}-3)}S^n\,. \end{aligned} \tag{27}$$

Again, the scaling for S *vs.* ε_q can be deduced from Eq. (27) by taking the limit $q \to q_{\rm c}$ ($\varepsilon_q \to 0$) and retaining on the right-hand side only the linear and the dominant nonlinear term with respect to S. For $\tilde{\gamma} > 5/2$, this yields

$$-\frac{4q_{\rm c}}{\sqrt{2\pi}}\frac{\tilde{\gamma}-2}{\tilde{\gamma}-5/2}m^{1/2}\varepsilon_q = 2\left(1-2q_{\rm c}\right)\left(\tilde{\gamma}-2\right)\left(\frac{m}{2}\right)^{\tilde{\gamma}-2} I(\tilde{\gamma})S^{2\tilde{\gamma}-5} - \sqrt{\frac{2}{\pi}}\frac{\left(1-2q_{\rm c}\right)\left(\tilde{\gamma}-2\right)}{3!}\frac{m^{3/2}}{2\tilde{\gamma}-7}S^2 + \ldots \tag{28}$$

For $\tilde{\gamma} > 7/2$, the dominant term on the right-hand side of Eq. (28) is that proportional to S^2; as a result, the order parameter S scales as $S \propto \varepsilon_q^{1/2}$ and the MV model on SF networks belongs to the standard MF universality class. In contrast, for $5/2 < \tilde{\gamma} < 7/2$, the dominant term on the right-hand side of Eq. (28) is that proportional to $S^{2\tilde{\gamma}-5}$ and the resulting scaling $S \propto \varepsilon_q^{\frac{1}{2(\tilde{\gamma}-5/2)}}$ is non-universal and depends on $\tilde{\gamma}$. Since the factors $1-2p$ cancel on both sides of Eq. (28), the expected scaling for S *vs.* ε_q does not depend on p.

The scaling for the magnetization M can be obtained from the equation

$$\begin{aligned} M &= N^{-1}\sum_{i=1}^{N}\langle s_i\rangle = \sum_k p_k\langle s_k\rangle \approx (1-2p)(1-2q)\sum_k p_k \mathrm{erf}\left(\sqrt{\frac{k}{2}}S\right) \\ &\approx \left(1-2p\right)\left(1-2q\right)A\int_m^\infty k^{-\tilde{\gamma}}\mathrm{erf}\left(\sqrt{\frac{k}{2}}S\right)\mathrm{d}k\,. \end{aligned} \tag{29}$$

(*cf.* Eq. (15)). Following the procedure applied to Eq. (24), the right-hand side of Eq. (29) can be written as a converging sum

$$M = 2^{-\tilde{\gamma}+2}\left(1-2q\right)AJ(\tilde{\gamma})S^{2\tilde{\gamma}-2} - 2m^{1-\tilde{\gamma}}\left(1-2q\right)A\sum_{n=0}^{\infty}\frac{\left(m/2\right)^{n/2}f_n}{n+1-\left(2\tilde{\gamma}-1\right)}S^n\,, \tag{30}$$

where, again, $f(u) = \mathrm{erf}(u)$, *etc.*, and $J(\tilde{\gamma}) = \int_0^\infty \frac{\hat{f}(u)}{u^{2\tilde{\gamma}-1}}\mathrm{d}u < 0$ with $\hat{f}(u) = f(u) - \sum_{n=0}^{m_0'-1} f_n u^n$, where m_0' is an integer number such that $m_0' < 2\tilde{\gamma}-1 < m_0'$. For $q \to q_{\rm c}$ and for $\tilde{\gamma} > 5/2$, the dominant term on the right-hand side of Eq. (30) is linear with respect to S. Thus, in the first approximation, there is $M \propto S$ and the magnetization M scales with ε_q in the same way as the order parameter S.

Due to the symmetry of Eq. (24) with respect to the exchange of q and p, it is obvious that for fixed q, the scaling of S and M with respect to ε_p is the same as that with respect to ε_q for fixed p.

It is interesting to compare the above results with the scaling of the magnetization of the Ising model on heterogeneous SF networks with the degree distribution (5) below the FM transition point [24–28]. In this case, the FM transition occurs for $\tilde{\gamma} > 3$ and the magnetization scales as $M \propto \varepsilon^{\frac{1}{\tilde{\gamma}-3}}$ for $3 < \tilde{\gamma} < 5$ and as $M \propto \varepsilon^{1/2}$ for $\gamma \geq 5$, where $\varepsilon = (T_{\mathrm{c}} - T)/T_{\mathrm{c}}$, and T_{c} denotes the critical temperature. These results were obtained using various methods of MF character, *e.g.*, heterogeneous MF theory, belief propagation algorithm and the replica method. It can be seen that the critical behavior of the MV and the Ising models on highly heterogeneous SF networks is different in the vicinity of the FM transition point. In particular, magnetization in both models can show a non-universal scaling with the critical exponent β depending on $\tilde{\gamma}$, but this scaling is different and occurs for different ranges of $\tilde{\gamma}$.

5. Comparison with Monte Carlo simulations

5.1. Methods of numerical analysis

For the purpose of MC simulations, SF networks are generated using the Configuration Model (CM) [29]. The algorithm starts with assigning to each node i, in a set of N nodes, a degree, *i.e.*, a random number k_i of ends of edges drawn from a given probability distribution p_k, with $m < k_i < N$ (the minimum degree of node is m, and the maximum one $N-1$), with the condition that the sum $\sum_i k_i$ is even. The network is completed by connecting pairs of the ends of edges chosen uniformly at random to make complete edges, respecting the preassigned sequence k_i and under the condition that multiple and self-connections are forbidden. SF networks generated from the CM are uncorrelated for $\tilde{\gamma} < 3$, and for $\tilde{\gamma} \geq 3$, they are correlated unless artificial constraints are imposed on the maximum degree of nodes [30].

Due to the symmetry of the model with respect to the exchange of the parameters q, p, resulting from the form of the spin-flip rate, Eq. (4), it is enough to study the FM transition as the parameter q is varied, with p fixed. MC simulations are performed using simulated annealing algorithm with random sequential updating of the agents' opinions: each simulation is started with high value of q corresponding to the PM phase and with random initial conditions, the parameter q is decreased in small steps toward zero, and for each intermediate value of q, time series of the instantaneous magnetization $\tilde{m} = N^{-1}\sum_{i=1}^{N} s_i$ are collected after initial transient. Next, the magnetization M, susceptibility χ and the fourth-order Binder cumulant U_4 are evaluated as functions of q

$$M(q) = [\langle|\tilde{m}|\rangle_t]_{\mathrm{av}} , \tag{31}$$

$$\chi(q) = N\left[\left(\left\langle\tilde{m}^2\right\rangle_t - \langle|\tilde{m}|\rangle_t^2\right)\right]_{\mathrm{av}} , \tag{32}$$

$$U_4(q) \;=\; \frac{1}{2}\left[3-\frac{\left\langle \tilde{m}^4\right\rangle_t}{\left\langle \tilde{m}^2\right\rangle_t^2}\right]_{\mathrm{av}} , \tag{33}$$

where $\langle\cdot\rangle_t$ denotes time average for a given realization of the SF network (usually over 2.5×10^4 MC simulation steps, each corresponding to updating N nodes) and $[\cdot]_{\mathrm{av}}$ denotes averaging over different realizations of the SF network with given N and $\tilde{\gamma}$; usually, results are averaged over 100–400 such realizations, depending on N ranging from 2×10^4 to 10^3, respectively. The above quantities are expected to obey FSS relations analogous to those valid for equilibrium systems on complex heterogeneous networks [27]

$$M \;=\; N^{-\beta/\nu} f_M\left(N^{1/\nu}(q-q_{\mathrm{c}})\right) , \tag{34}$$

$$\chi \;=\; N^{\gamma/\nu} f_\chi\left(N^{1/\nu}(q-q_{\mathrm{c}})\right) , \tag{35}$$

$$q_{\mathrm{c}}-q^\star(N) \;\propto\; N^{-1/\nu} , \tag{36}$$

where $q^\star(N)$ denotes the value of q for which the susceptibility χ of the model on SF network with N nodes has a maximum value.

The critical value q_{c} for the FM transition can be obtained from the intersection point of the Binder cumulants for different sizes N of the SF network [31]. Next, from Eqs. (34), (35), the exponents β/ν and γ/ν, respectively, can be determined. Furthermore, Eq. (36) can be used to calculate the exponent $1/\nu$; as a result, the critical exponents β and γ can be finally evaluated. Eventually, it is verified if the obtained exponents fulfil the hyperscaling relation

$$2\frac{\beta}{\nu}+\frac{\gamma}{\nu}=D_{\mathrm{eff}} , \tag{37}$$

where the effective dimension $D_{\mathrm{eff}}=1$ is expected in the case of systems on complex networks (and, consequently, on SF networks) which do not have any particular spatial dimension [27].

5.2. Results and discussion

In Fig. 1, exemplary curves of the magnetization M, susceptibility χ and Binder cumulant U_4 *vs.* q are shown obtained from MC simulations of the MV model with $p=0$ on SF networks with $\tilde{\gamma}=3.25$ and different numbers of nodes N. Second-order FM transition is evidenced by smooth increase of the magnetization, occurrence of the maximum of the susceptibility and monotonic growth of the Binder cumulants for decreasing q. The critical value q_{c} can be determined with high accuracy from the intersection point of the Binder cumulants for various system sizes N. In a similar way, the critical values q_{c} were determined from MC simulations of the MV model with different values of p on SF networks with high mean degree of nodes and different degree distributions. The obtained dependence of q_c on p (Fig. 2) agrees well with the prediction of the heterogeneous MF theory, Eq. (19).

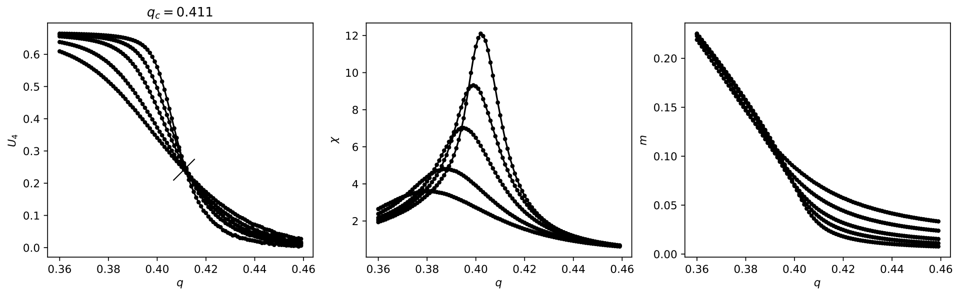

Fig. 1. The Binder cumulant U_4, the susceptibility χ and the magnetization M *vs.* q for the MV model with $p = 0$ on SF networks with $\tilde{\gamma} = 3.25$ and $N = 10^3$, 2×10^3, 5×10^3, 10^4, 2×10^4 (in all cases, curves with smaller maximum curvature correspond to smaller N).

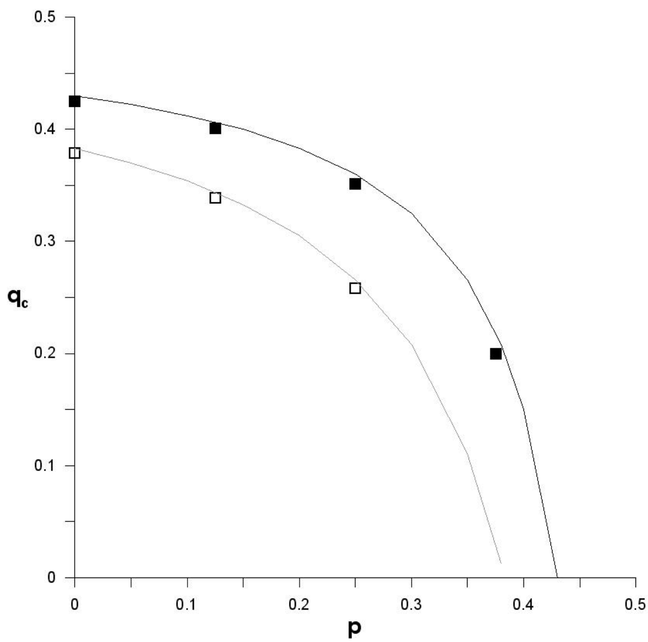

Fig. 2. The critical value q_c *vs.* p for the MV model on SF networks with minimum degree $m = 20$, symbols: results obtained from MC simulations (filled for $\tilde{\gamma} = 3$, empty for $\tilde{\gamma} = 5$), solid lines: theoretical result obtained in the MF approximation (black for $\tilde{\gamma} = 3$, gray for $\tilde{\gamma} = 5$), Eq. (19).

In Fig. 3, exemplary results of the FSS analysis are shown for the MV model with $p = 0$ on SF networks with $\tilde{\gamma} = 3.25$. The magnetzation M and susceptibility χ at $q = q_{\rm c}$ as well as the location of the maximum of the susceptibility $q^{\star}(N) - q_{\rm c}$ exhibit power scaling as functions of the number of nodes N, in accordance with Eqs. (34), (35), (36), from which the exponents β/ν, γ/ν, $1/\nu$ can be determined using the least squares fit method (Fig. 3). Accuracy of the FSS analysis is confirmed by the observed coincidence for different system sizes N of the curves M *vs.* q and χ *vs.* q rescaled according to Eqs. (34) and (35), respectively (Fig. 4).

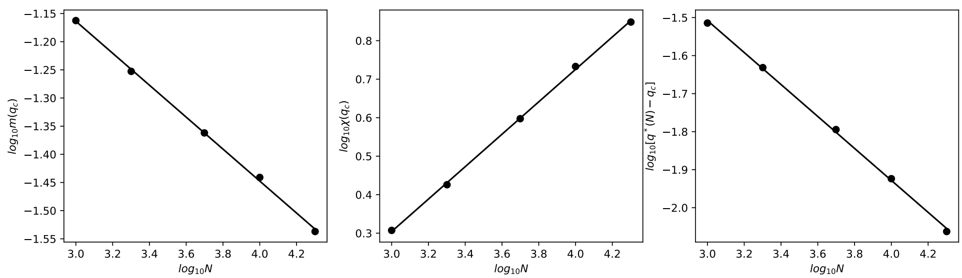

Fig. 3. Log–log plots of the curves $M\,(q_{\rm c})$, $\chi\,(q_{\rm c})$ and $q^{\star}(N) - q_{\rm c}$ *vs.* N for the MV model with $p = 0$ on SF networks with $\tilde{\gamma} = 3.25$ and $N = 10^3$, 2×10^3, 5×10^3, 10^4, 2×10^4.

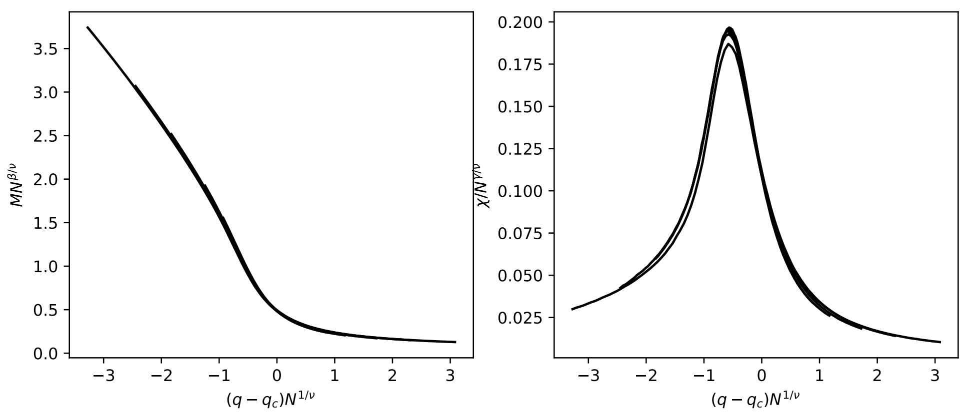

Fig. 4. Rescaled magnetization $MN^{\beta/\nu}$ and susceptibility $\chi N^{-\gamma/\nu}$ *vs.* $(q - q_{\rm c})\,N^{1/\nu}$ for the MV model with $p = 0$ on SF networks with $\tilde{\gamma} = 3.25$ and $N = 10^3$, 2×10^3, 5×10^3, 10^4, 2×10^4.

The exponents β/ν, γ/ν, $1/\nu$ obtained from the FSS analysis for the MF model with $p = 0$ on SF networks with different degree distributions and minimum degrees are summarized in Tables I–III. In all cases, the hyperscaling relation (37) is fulfilled with good accuracy. Concerning the critical exponent β for the magnetization below the FM transition point, for given degree distribution of the SF network, it does not show any characteristic behavior as a function of the mean degree: for highly heterogeneous network with $\tilde{\gamma} = 3$, it increases with the minimum degree m (Table I), while for less heterogeneous network with $\tilde{\gamma} = 5$, it decreases with m (Table II). However, in both cases for large m (large $\langle k \rangle$), it approaches the values predicted in the heterogeneous MF approximation, Sec. 4.2, $\beta = 1/\left[2\left(\tilde{\gamma} - 5/2\right)\right] = 1$ for $\tilde{\gamma} = 3 < 7/2$ and $\beta = 1/2$ for $\tilde{\gamma} = 5 > 7/2$. In fact, for the MV model with $p = 0$ on SF networks with high mean degree of nodes, the value of the critical exponent β obtained from the FSS analysis agrees well with that evaluated in the heterogeneous MF approximation for a wide range of $\tilde{\gamma}$ (Table III and Fig. 5). Only for $\tilde{\gamma} = 2.75 < 3$, the numerical and theoretical values of β differ noticeably; this difference can be attributed to the correlations between the degrees of nodes which inevitably occur in SF networks with diverging second moment of the degree distribution generated from the CM model.

TABLE I

The critical value q_c, exponents β/ν, γ/ν, $1/\nu$, effective dimension D_{eff} and the critical exponent β for the MV model with $p = 0$ on SF network with $\tilde{\gamma} = 3.0$ and different minimum degree m.

m	q_c	β/ν	γ/ν	$1/\nu$	D_{eff}	β
2	0.157	0.207(20)	0.633(27)	0.404(27)	1.047(67)	0.512(84)
5	0.332	0.291(16)	0.439(22)	0.340(3)	1.022(54)	0.856(55)
10	0.389	0.294(9)	0.409(13)	0.353(5)	0.998(31)	0.832(37)
20	0.425	0.311(7)	0.376(10)	0.323(6)	0.998(24)	0.962(40)
50	0.453	0.317(3)	0.363(6)	0.322(4)	0.997(12)	0.984(22)

TABLE II

The critical value q_c, exponents β/ν, γ/ν, $1/\nu$, effective dimension D_{eff} and the critical exponent β for the MV model with $p = 0$ on SF network with $\tilde{\gamma} = 5.0$ and different minimum degree m.

m	q_c	β/ν	γ/ν	$1/\nu$	D_{eff}	β
2	0.022	0.263(9)	0.488(6)	0.418(28)	1.015(24)	0.629(64)
5	0.239	0.256(2)	0.501(2)	0.510(7)	1.013(6)	0.502(11)
10	0.324	0.267(1)	0.476(2)	0.498(6)	1.011(4)	0.537(8)
20	0.379	0.256(2)	0.484(4)	0.533(9)	0.996(8)	0.480(12)
50	0.425	0.274(3)	0.456(5)	0.508(6)	1.005(4)	0.540(12)

TABLE III

The critical value q_c, exponents β/ν, γ/ν, $1/\nu$, effective dimension $D_{\rm eff}$, the critical exponent β and its theoretical value (Sec. 4.2) for the MV model with $p = 0$ on SF networks with different $\tilde{\gamma}$ and minimum degree $m = 20$.

$\tilde{\gamma}$	q_c	β/ν	γ/ν	$1/\nu$	$D_{\rm eff}$	β	$\beta_{\rm theor}$
2.75	0.448	0.356(6)	0.290(10)	0.221(2)	1.002(22)	1.606(42)	2.0
3.0	0.425	0.311(7)	0.376(10)	0.323(6)	0.998(24)	0.962(40)	1.0
3.25	0.411	0.284(5)	0.421(6)	0.420(7)	0.988(13)	0.675(19)	0.667
3.5	0.403	0.287(3)	0.424(4)	0.442(6)	0.998(10)	0.649(16)	0.5
4.0	0.391	0.265(3)	0.461(4)	0.504(7)	0.992(10)	0.527(13)	0.5
5.0	0.379	0.256(2)	0.484(4)	0.533(9)	0.996(8)	0.480(12)	0.5
6.0	0.373	0.258(2)	0.486(3)	0.523(4)	1.001(7)	0.492(8)	0.5

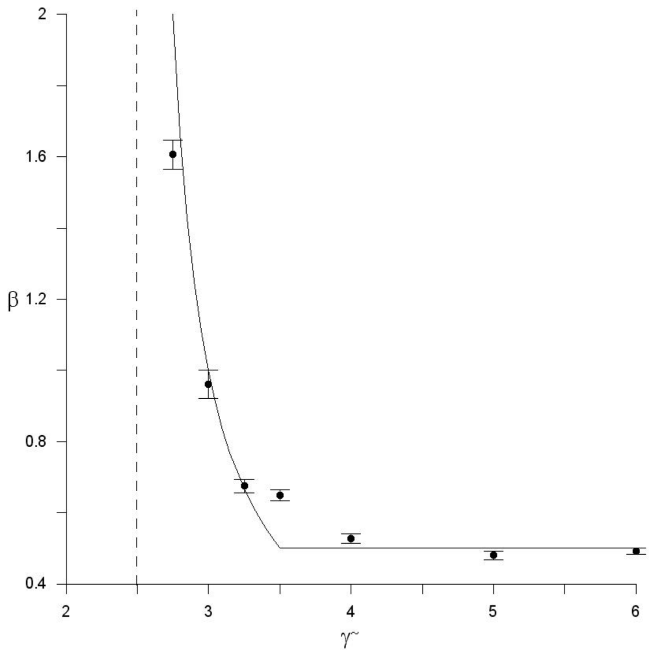

Fig. 5. The critical exponent β *vs.* $\tilde{\gamma}$ for the MV model with $p = 0$ on SF network with minimum degree $m = 20$, symbols: results obtained from MC simulations and FSS analysis, solid line: theoretical result obtained in the MF approximation (*cf.* Table III).

The observed increase of the exponent $1/\nu$ with $\tilde{\gamma}$ for fixed high mean degree of nodes up to $1/\nu \approx 0.5$ (Table III) agrees with that reported in Ref. [11]. As a result, with increasing $\tilde{\gamma}$, the exponent β/ν decreases to *ca.* 0.25 and γ/ν increases to *ca.* 0.5, *i.e.*, the critical exponent γ reaches the value $\gamma \approx 1$. Similar values were reported for the MV model on Erdös–Rényi graphs [6]; this can be expected since less heterogeneous SF networks resemble more regular random graphs. Thus, the MV model on weakly heterogeneous SF networks belongs to the standard MF universality class characterized by the critical exponents $\beta = 1/2$, $\gamma = 1$, the same as for the Ising model. However, theoretical arguments from Sec. 4.2 and numerical results in Table III and Fig. 5 suggest that the critical behavior of the MV model on SF networks is universal for $\tilde{\gamma} > 7/2$, while that of the Ising model only for $\tilde{\gamma} > 5$. Thus, both models belong to the same standard MF universality class only for $\tilde{\gamma} > 5$.

In Tables IV–V, the exponents β/ν, γ/ν, $1/\nu$ obtained from the FSS analysis are summarized for the MV model on SF networks with high ($\tilde{\gamma} = 3$, Table IV) and lower ($\tilde{\gamma} = 5$, Table V) heterogeneity and high mean degree of nodes, for different values of the independence parameter p. As can be seen, these exponents and thus also the critical exponent β with reasonable accuracy do not depend on p, and the latter exponent is always close to the theoretical value $\beta = 1$ for $\tilde{\gamma} = 3$ and $\beta = 1/2$ for $\tilde{\gamma} = 5$, as expected (*cf.* Sec. 4.2).

TABLE IV

The critical value q_c, exponents β/ν, γ/ν, $1/\nu$, effective dimension D_{eff} and the critical exponent β for the MV model with different p on SF network with $\tilde{\gamma} = 3.0$ and minimum degree $m = 20$.

p	q_c	β/ν	γ/ν	$1/\nu$	D_{eff}	β
0.0	0.425	0.311(7)	0.376(10)	0.323(6)	0.998(24)	0.962(40)
0.125	0.401	0.317(7)	0.366(11)	0.314(6)	1.001(25)	1.010(42)
0.25	0.351	0.315(7)	0.370(11)	0.317(5)	1.000(25)	0.995(38)
0.375	0.2	0.313(7)	0.373(10)	0.324(5)	0.999(24)	0.965(37)

TABLE V

The critical value q_c, exponents β/ν, γ/ν, $1/\nu$, effective dimension D_{eff} and the critical exponent β for the MV model with different p on SF network with $\tilde{\gamma} = 5.0$ and minimum degree $m = 20$.

p	q_c	β/ν	γ/ν	$1/\nu$	D_{eff}	β
0.0	0.379	0.256(2)	0.484(4)	0.533(9)	0.996(8)	0.480(12)
0.125	0.339	0.262(2)	0.475(4)	0.514(7)	1.000(8)	0.510(11)
0.25	0.258	0.255(3)	0.484(5)	0.518(7)	0.994(11)	0.492(12)

6. Conclusions

The MV model on complex SF networks with the degree distribution $p_k \propto k^{\tilde{\gamma}}$ was investigated analytically in the heterogeneous MF approximation, and numerically by MC simulations and FSS analysis. In the model, it was taken into account that the agents can make decisions independently with probability controlled by the parameter p and otherwise take into account the majority opinion in their neighborhood which they then follow with probability controlled by another parameter q. For $\tilde{\gamma} > 5/2$, the model can show the FM transition as one of the above-mentioned parameters is decreased, with the other fixed. Critical values of the parameters q_{c}, p_{c} for the FM transition with fixed p and q, respectively, were evaluated in the heterogeneous MF approximation which agree with those obtained from MC simulations. Critical behavior of the magnetization was studied in the heterogeneous MF approximation. It was shown that below the FM transition point for $5/2 < \tilde{\gamma} < 7/2$, the scaling of the magnetization is non-universal, with the critical exponent $\beta = 1/\left[2\left(\tilde{\gamma} - 5/2\right)\right]$, while for $\tilde{\gamma} > 7/2$, it becomes universal, with $\beta = 1/2$. This result was confirmed by MC simulations and FSS analysis of the model on SF networks with high mean degree of nodes. Moreover, estimates of the remaining critical exponents γ, ν from MC simulations and FSS analysis show that for $\tilde{\gamma} > 7/2$, the MV model on SF networks belongs to the standard MF universality class. Taking into account the well-known critical properties of the Ising model on SF networks, it may be concluded that both models belong to the same standard MF universality class for $\tilde{\gamma} > 5$.

Appendix

In Ref. [14], the MV model with independent agents was investigated on two-dimensional regular lattices, and by MC simulations the critical value of the independence parameter p_{c} for the occurrence of the FM transition for fixed q was found to follow the empirical relation

$$\tilde{p}_{\mathrm{c}} = 2p_{\mathrm{c}} = p_{\mathrm{c}}(q) = \frac{aq + b}{cq + d}\,, \tag{38}$$

with $a = 1$, $b = 0.075$, $c = 0.973$, $d = -0.5$. It is interesting to note that the relation p_{c} *vs.* q in the same functional form follows from Eq. (20) with $\langle k \rangle = 4$ and $\langle k^{3/2} \rangle = 8$ for the two-dimensional regular lattice, which yields $a = 1$, $b = -0.187$, $c = 1$, $d = -0.5$ in Eq. (38). Thus, in this case, the heterogeneous MF approximation overestimates p_{c} for given q (*e.g.*, it yields $p_{\mathrm{c}}(0) = 0.374$ in comparison with $p_{\mathrm{c}}(0) \approx 0.15$ obtained from MC simulations in Ref. [14]), but the qualitative relation between p_{c} and q is correct.

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