RAPIDITY GAPS AND ANCESTRY* **

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The recently discovered correspondence between the distribution of rapidity gaps in electron–nucleus diffractive processes and the statistics of the height of genealogical trees in branching random walks is reviewed. In addition, a new comparison of numerical solutions of exact equations for diffraction on the one hand, and for ancestry on the other hand, both established in the framework of the color dipole model, is presented.

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1. Rapidity gap distribution in deep-inelastic scattering

In the scattering of electrons off protons at very high energies[1], a particularly striking — and *a priori* surprising — phenomenon was discovered experimentally: *hard diffraction*. In a significant proportion of the events (about 10% overall at the DESY-HERA collider), the proton left the collision unaltered, while in the forward region of the scattered electron, a hadronic system was observed, as a result of the dissociation of the virtual photon mediating the interaction (see Fig. 1). Diffractive events may be labeled with the size of the region void of particles surrounding the scattered proton, which can be characterized by a Lorentz-invariant *rapidity gap* variable y_0. The latter fluctuates from event-to-event between (almost) zero and the maximum available rapidity $Y = \ln \hat{s}/Q^2$, where $\hat{s}$ is the squared center-of-mass energy of the γ^*-proton/nucleus subreaction, and Q the virtuality of the photon. What has been observed in high-energy electron–proton scattering is also expected in electron–nucleus collisions at a future Electron–Ion Collider (EIC).

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[1] For the background on all aspects of high-energy scattering, see the textbook of Ref. [1].

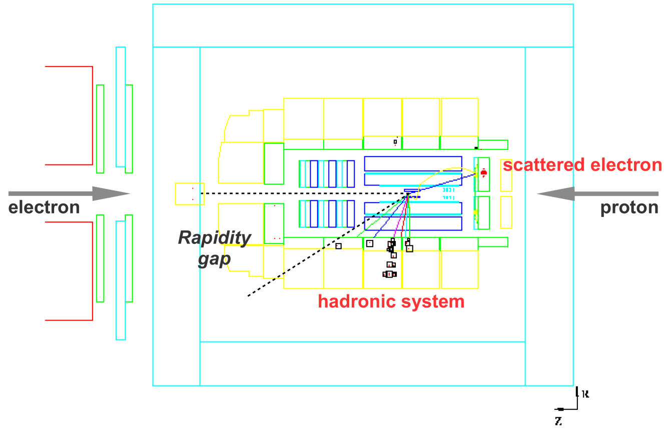

Fig. 1. Diffractive dissociation event recorded in the H1 detector. The highly-energetic proton (entering from the right) interacts elastically, picks a small transverse momentum compared to its longitudinal momentum and, therefore, does not leave any track in the detector. The virtual photon instead is converted into a hadronic system. The angular sector between the momentum of the scattered proton and the produced hadronic system void of any activity is the rapidity gap.

Some time ago, an equation for the distribution of rapidity gaps was in this context rigorously established by Kovchegov and Levin (KL) [2]. However, solving it analytically remains a formidable challenge. They did not address directly deep-inelastic scattering, but instead onium–nucleus scattering, which is straightforwardly related to the former when the interaction between the electron and the nucleus is mediated by a *longitudinally-polarized* virtual photon.

Let us consider an onium of size r scattering off a big nucleus. In the KL formulation, the distribution of the rapidity gaps is the solution of a system of two equations. The first one is the Balitsky–Kovchegov (BK) equation for the rapidity evolution of the forward elastic S-matrix element[2]. Introducing the notation $\bar{\alpha} \equiv \alpha_{\rm s} N_{\rm c}/\pi$, the BK equation reads

$$\partial_y S(r,y) = \bar{\alpha} \int \frac{{\rm d}^2 r'}{2\pi} \frac{r^2}{r'^2 (r-r')^2} \left[S(r',y)\, S(r-r',y) - S(r,y) \right] . \quad (1)$$

The initial condition is given by *e.g.* the McLerran–Venugopalan (MV) model, $S(r, y=0) = e^{-\frac{r^2 Q_{\rm MV}^2}{4} \ln(e + 4/r^2 \Lambda_{\rm QCD}^2)}$, with $Q_{\rm MV}$ the saturation mo-

[2] The (dimensionless) total, elastic and inelastic cross sections *per impact parameter* may be derived from S, which is essentially real at high energy: $\sigma_{\rm tot} = 2(1-S)$, $\sigma_{\rm el} = (1-S)^2$, $\sigma_{\rm in} = \sigma_{\rm tot} - \sigma_{\rm el} = 1 - S^2$. These formulas show, in particular, that the elastic cross section is maximum (and equal to the inelastic one) when $S = 0$.

mentum of the nucleus. $1/Q_{\rm MV}$ can be interpreted as the dipole size above which the scattering occurs with unit probability [*i.e.* $S(r\gg 1/Q_{\rm MV},0)\ll 1$]. The rapidity gap distribution is deduced from an auxiliary function $S_2(r,\tilde{y})$ which also obeys the BK equation[3]

$$\partial_{\tilde{y}} S_2\left(r,\tilde{y}\right) = \bar{\alpha}\int \frac{\mathrm{d}^2 r'}{2\pi}\frac{r^2}{r'^2\left(r-r'\right)^2}\left[S_2\left(r',\tilde{y}\right)S_2\left(r-r',\tilde{y}\right)-S_2\left(r,\tilde{y}\right)\right]\,, \tag{2}$$

with the initial condition $S_2(r,\tilde{y}=0)=[S(r,y_0)]^2$. In terms of S_2, the gap distribution then reads

$$\frac{\mathrm{d}\sigma_{\rm diff}(y_0|r,Y)}{\mathrm{d}y_0} = \left.\frac{\partial}{\partial\tilde{y}}\right|_{\tilde{y}=Y-y_0} S_2(r,\tilde{y})\,. \tag{3}$$

The work presented here may be viewed as an effort to find a solution to the KL set of equations (1), (2). However, instead of trying to solve it brute force, which is technically extremely challenging, we develop a picture of diffractive scattering from which, what we believe, should be the asymptotics of the KL equation (almost) straightforwardly follow and which points to a deep link with ancestry problems in branching random walks. The present write-up shortly summarizes the papers in Refs. [3, 4] before presenting a new numerical comparative study of exact equations for diffraction and ancestry in the dipole model (see Sec. 3.2 below).

2. Picture of onium–nucleus scattering

2.1. Total cross section in the onium and nucleus restframes

In the onium restframe, the nucleus appears in a highly-evolved and occupied state, while the dominant state of the onium is a bare quark–antiquark pair. Event-by-event fluctuations are negligible; S may be interpreted as the "transparency" of the boosted nucleus.

In the nucleus restframe instead, the whole evolution is in the onium, which appears typically as a set of many gluons (represented by dipoles in the large number-of-color limit [5]), whose detailed content strongly fluctuates from event-to-event. For a scattering to occur, there should be at least one gluon in this set which has a transverse momentum of the order of $Q_{\rm MV}$, so that the whole state has a non-negligible probability to scatter with the nucleus. In this context, $1-S$ can be interpreted as the probability that the Fock state of the onium contains at least one gluon with a transverse momentum of that magnitude.

[3] Equations (1), (2) also follow quite straightforwardly from the Good–Walker picture, see Ref. [3].

2.2. Diffractive cross section in the y_0-frame

Let us now choose a frame in which the nucleus is boosted to rapidity y_0 and the onium to rapidity $\tilde{y}_0 = Y - y_0$. In order to have a diffractive event exhibiting a gap y_0 with unit probability, one needs at least one gluon in the Fock state of the onium whose transverse momentum is smaller than the saturation scale at rapidity y_0, $Q_s(y_0)$[4]. Indeed, this condition makes sure that the elastic interaction cross section of the onium is significant. The diffractive cross section $d\sigma_{diff}/dy_0$ is tantamount to this very probability. A straightforward calculation leads to an elegant formula for the latter, once normalized to the total cross section

$$\frac{1}{\sigma_{tot}}\frac{d\sigma_{diff}}{dy_0} = \text{const} \times \left[\frac{\bar{\alpha}Y}{\bar{\alpha}y_0(\bar{\alpha}Y - \bar{\alpha}y_0)}\right]^{3/2} . \tag{4}$$

The overall numerical constant, of the order of unity, cannot be determined within the present approach. This formula is actually only valid in the so-called *scaling region*, defined by the following constraints on the parameters: $1 \ll \ln r^2 Q_s^2(Y) \ll \sqrt{\chi''(\gamma_0)\bar{\alpha}Y}$.

3. Ancestry

3.1. Height of genealogical trees in branching random walks

It has been known for some time that dipole evolution is a peculiar branching random walk [6]. One of the main results of Refs. [3, 4] is the surprising observation that the structure of the branches may be directly related to an observable in high-energy physics.

Boosting a bare onium of size r by one unit in the rapidity $\tilde{y}$ opens the phase space for quantum fluctuations in the form of additional gluons populating its Fock state. A one-gluon emission by the onium may be interpreted as the splitting of a color dipole into two dipoles, of different sizes. Upon a further boost, each of these two dipoles may split independently through the same process. Thus, one understands that QCD evolution is a branching process of dipoles in rapidity with a random walk in the sizes of the latter[5].

Now boost to rapidity Y takes *e.g.* the two largest dipole in the Fock state and track their first common ancestor. According to Ref. [7], the rapidity y_0 at which the ancestor branches is distributed as

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[4] Denoting by $\chi(\gamma)$ the eigenvalue of the linearized BK equation about $S \sim 1$ corresponding to the eigenfunction $1 - S = r^{2\gamma}$ and by γ_0 the solution of the equation $\chi(\gamma_0) = \gamma_0\chi'(\gamma_0)$, one has $Q_s^2(y_0) = Q_{MV}^2 e^{\bar{\alpha}y_0\chi'(\gamma_0)}/(\bar{\alpha}y_0)^{3/2\gamma_0}$.

[5] The relevant scale for the dipole sizes is logarithmic, and the relevant evolution variable is the scaled rapidity $\bar{\alpha}Y$. Actually, the process is diffusive only if one looks at a fixed impact parameter, but this is what turns out to be relevant here.

$$p(y_0|r,Y) = c_p\left[\frac{\bar{\alpha}Y}{\bar{\alpha}y_0(\bar{\alpha}Y-\bar{\alpha}y_0)}\right]^{3/2} \quad \text{with } c_p = \frac{1}{\bar{\gamma}}\frac{1}{\sqrt{2\pi\chi''(\gamma_0)}}\,. \qquad (5)$$

The value of $\bar{\gamma}$ depends on which dipoles are picked: In the present case, $\bar{\gamma}$ coincides with γ_0. Formula (5) was actually not established in the peculiar context of dipole evolution of interest for particle physics, but was argued to apply to a wide class of branching random walks. We also expect it to be correct (up to the overall numerical factor) for related quantities, such as the rapidity distribution of the common ancestor of *all dipoles larger than some given (large enough) size* $1/Q_{\rm MV}$[6].

Equation (5) was found by *assuming* that the common ancestor was an unusually large object generated around the rapidity $\tilde{y}_0 = Y - y_0$ in the evolution of the onium [7]. This is exactly the same mechanism as in the case of the diffraction problem (see Sec. 2.2). Hence, the two problems are intimately related: up to the overall normalization, which is determined in the case of the genealogies but not in the case of diffraction, $(1/\sigma_{\rm tot})({\rm d}\sigma_{\rm diff}/{\rm d}y_0)$ corresponds to $p(y_0|r,Y)$.

In order to check quantitatively this correspondence between diffraction and ancestry, we have established exact equations for ancestry and have compared their numerical solutions to those for diffraction.

3.2. Ancestry equation for dipoles and its numerical solution

The distribution $p_>(y_0|r,Y)$ of the rapidity at which the first common ancestor of all dipoles larger than $1/Q_{\rm MV}$ (or, alternatively, of a set of dipoles randomly picked among the dipoles present in the Fock state at rapidity Y with some probability $T(r)$) first splits obeys the following equation:

$$\partial_y p_>(y_0|r,y) = \bar{\alpha}\int\frac{{\rm d}^2r'}{2\pi}\frac{r^2}{r'^2\,(r-r')^2} \times\left[p_>\left(y_0|r',y\right)S\left(r-r',y\right)+S\left(r',y\right)p_>\left(y_0|r-r',y\right)-p_>(y_0|r,y)\right]\,, \quad (6)$$

where S solves Eq. (1) with the initial condition $S(r,0) = 1 - T(r)$[7]. The initial condition for $p_>$ reads

$$p_>(y_0|r,y_0) = \bar{\alpha}\int\frac{{\rm d}^2r'}{2\pi}\frac{r^2}{r'^2\,(r-r')^2}\left[1-S\left(r',y_0\right)\right]\left[1-S\left(r-r',y_0\right)\right]\,. \quad (7)$$

[6] This holds true if r, $\bar{\alpha}Y$ and $Q_{\rm MV}$ are such that one is in the scaling region defined at the end of Sec. 2.2.

[7] If one wanted to consider the common ancestor of all dipoles larger than $1/Q_{\rm MV}$, then one would use as an initial condition $T(r,0) = \theta(\ln r^2Q^2_{\rm MV})$. In our calculation, $T = 1 - S$, where S is given by the MV model, which is a bit less sharp than the θ function.

The numerical solutions of the equations for $p_>$ and for $(1/\sigma_{\rm tot})(d\sigma_{\rm diff}/dy_0)$ are shown in Fig. 2: Both are in very good agreement with Eq. (4), with an overall constant of the order of 1. Trying to understand analytically this constant is one of our current goals.

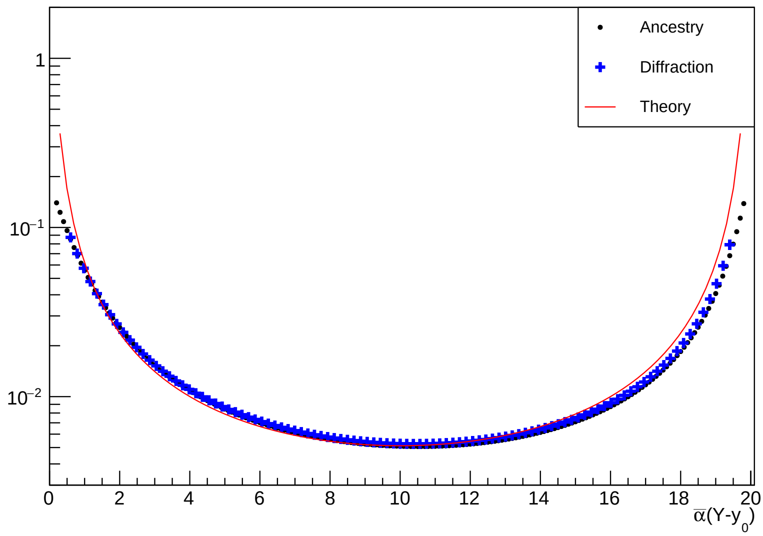

Fig. 2. Numerical solution of the equation for $(1/\sigma_{\rm tot})(d\sigma_{\rm diff}/dy_0)$ (labeled "Diffraction"; from Ref. [4]) and for $p_>$ ("Ancestry"; from Ref. [8]) as a function of $\bar{\alpha}(Y-y_0)$, with the parameters set to $\bar{\alpha}Y = 20$ and $rQ_{\rm MV} = 4\times 10^{-21}$. The continuous line has been generated from the analytical formula (5) with $\bar{\gamma} = 1$.

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