

ASYMPTOTICALLY CONSERVED QUANTITIES OF THE GRAVITATIONAL CHARGES ASSOCIATED WITH (3+1) DECOMPOSITION OF CYK TENSORS*

JACEK JEZIERSKI, TOMASZ SMOLKA

Department of Mathematical Methods in Physics
University of Warsaw, Pasteura 5, 02-093 Warszawa, Poland

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The conformal Yano–Killing (CYK) tensor enables one to construct a conserved quantity (charge) as a two-dimensional surface integral in four-dimensional spacetime. The aim of the paper is to present a (3+1) decomposition of CYK tensor and discuss the splitting of CYK charge on Cauchy surface.

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1. Survey on CYK tensors and associated charges

Let M be a four-dimensional manifold with a Lorentzian metric $g_{\mu\nu}$. The covariant derivative associated with the Levi-Civita connection will be denoted by ∇ or just by “;”. “,” means directional (partial) derivative. By $T_{\dots(\mu\nu)\dots}$, we will denote the symmetric part and by $T_{\dots[\mu\nu]\dots}$ the skew-symmetric part of tensor $T_{\dots\mu\nu\dots}$ with respect to indices μ and ν (analogous symbols will be used for more indices).

Definition 1 *A skew-symmetric tensor field (two-form) $Q_{\mu\nu}$ is a conformal Yano–Killing (CYK) tensor for the metric g iff $Q_{\lambda\kappa;\sigma} + Q_{\sigma\kappa;\lambda} = \frac{2}{n-1} (g_{\sigma\lambda} Q^\nu{}_{\kappa;\nu} + g_{\kappa\lambda} Q_\sigma{}^\mu{}_{;\mu})$.*

CYK tensors are a generalization of conformal Killing vectors to two-forms. The set of equations in Definition 1 is overdetermined. The solutions exist mainly for type D spacetimes. Properties and detailed information can be found in [2] and in the references therein. In the context of further research, the following property is important:

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Theorem 1 (Hodge duality) *Let $g_{\mu\nu}$ be a metric of a four-dimensional differential manifold M . $*$ denotes the Hodge duality¹. A skew-symmetric tensor $Q_{\mu\nu}$ is a CYK tensor of the metric $g_{\mu\nu}$ iff its dual $*Q_{\mu\nu}$ is a CYK tensor of this metric.*

The conserved quantities are constructed from the spin-2 tensor field² $W_{\alpha\beta\mu\nu}$. $W_{\mu\nu\alpha\beta}$ is skew-symmetric both in the first and the second pair of indices, and to both of them the Hodge star can be applied. We denote

$$*W_{\mu\nu\alpha\beta} = \frac{1}{2}\varepsilon_{\mu\nu}{}^{\rho\sigma}W_{\rho\sigma\alpha\beta}, \quad W^*{}_{\mu\nu\alpha\beta} = \frac{1}{2}W_{\mu\nu\rho\sigma}\varepsilon^{\rho\sigma}{}_{\alpha\beta}. \quad (1)$$

The symmetries of spin-2 tensor provide $*W = W^*$, $*(W) = *W^* = -W$. Let $W_{\mu\nu\alpha\beta}$ be a spin-2 field and $Q_{\mu\nu}$ be a CYK tensor. Let us denote by F the following two-form:

$$F_{\mu\nu}(W, Q) := W_{\mu\nu\lambda\kappa}Q^{\lambda\kappa}. \quad (2)$$

The following formula is satisfied:

$$\nabla_\nu F^{\mu\nu}(W, Q) = 0. \quad (3)$$

Theorem 1 and Eq. (1) enable one to observe similarly

$$*F_{\mu\nu} = *W_{\mu\nu\lambda\kappa}Q^{\lambda\kappa} = W_{\mu\nu\lambda\kappa}(*Q)^{\lambda\kappa}, \quad \nabla_\nu(*F)^{\mu\nu}(W, Q) = 0. \quad (4)$$

Let V be a three-volume and ∂V its boundary. Formula (3) implies³

$$\int_{\partial V} F^{\mu\nu}(W, Q) d\sigma_{\mu\nu} = \int_V \nabla_\nu F^{\mu\nu}(W, Q) d\Sigma_\mu = 0.$$

In this sense, $Q_{\mu\nu}$ defines a charge related to the spin-2 field W ⁴

$$C(W, Q) := \int_S F^{\mu\nu}(W, Q) d\sigma_{\mu\nu}. \quad (5)$$

¹ Hodge duality (Hodge star) is given by $*\omega_{\alpha\beta} := \frac{1}{2}\varepsilon_{\alpha\beta}{}^{\mu\nu}\omega_{\mu\nu}$, where $\varepsilon_{\alpha\beta\mu\nu}$ is the skew-symmetric Levi-Civita tensor.

² Spin-2 tensor field fulfills: $W_{\alpha\beta\mu\nu} = W_{\mu\nu\alpha\beta} = W_{[\alpha\beta][\mu\nu]}$, $W_{\alpha[\beta\mu\nu]} = 0$, $g^{\alpha\mu}W_{\alpha\beta\mu\nu} = 0$, $\nabla_{[\lambda}W_{\mu\nu]\alpha\beta} = 0$.

³ Symbols $d\sigma_{\mu\nu}$ and $d\Sigma_\mu$ can be defined in the following way: if Ω stands for the volume form of the manifold M , then $d\sigma_{\mu\nu} := (\partial_\mu \wedge \partial_\nu) \lrcorner \Omega$, $d\Sigma_\mu := \partial_\mu \lrcorner \Omega$.

⁴ The flux of $F^{\mu\nu}$ through any two-dimensional closed surfaces S_1 and S_2 is the same as long as we are able to find a three-volume V between them (*i.e.* there exists V such that $\partial V = S_1 \cup S_2$).

2. (3+1) decomposition of CYK tensor for the Minkowski spacetime. Instant charges

The aim of this section is to present the decomposition of CYK tensor and associated conserved quantities for the Minkowski spacetime. In [3], the following decomposition of CYK tensor for the Minkowski spacetime has been proved:

Lemma 1 *Each CYK tensor in the Minkowski spacetime can be expressed in the following way:*

$$Q = a(t)\mathcal{T}_0 \wedge X + b(t) * (\mathcal{T}_0 \wedge Y), \quad (6)$$

where X, Y are (three-dimensional) conformal Killing fields; $a(t), b(t)$ are quadratic polynomials of a single indeterminate t .

The basis of the space of solutions for the equations, given in Definition 1, (i.e. the basis of CYK tensors) in the Minkowski spacetime is twenty-dimensional. This is a maximal possible dimension of space of CYK solutions for four-dimensional spacetime.

Spin-2 tensor can be splitted into well-known gravito-electro-magnetic tensors⁵. Let us consider how charge (5) is related with an instant charge (8) on Cauchy surface $\Sigma_t : t = \text{const.}$ We need to introduce the following conserved quantity:

Definition 2 (instant charge) *Consider a given spatial hypersurface Σ equipped with a conformally flat, Riemannian metric γ . Let A be a two-dimensional closed surface, embedded in Σ . We can define the following instant charges:*

$$I(E, X) := \int_A E^i_j X^j dS_i, \quad I(B, X) := \int_A B^i_j X^j dS_i, \quad (8)$$

where X^j is a conformal Killing vector field (CKV).

Instant charge does not depend on the choice of the surface A iff the three-dimensional divergence of gravito-electric (gravito-magnetic) tensor vanishes. The relation between four-dimensional CYK conserved quantity and CYK charge is as follows. Using (4), (5), (6), and decomposition into magnetic and electric part, we can prove

⁵ Let n^μ be a normed vector perpendicular to the foliation Σ_t . A spin-2 field on Σ_t can be equivalently described by two symmetric and traceless tensors: the electric and the magnetic part of Weyl tensor. Electric (magnetic) part

$$E_{\alpha\beta} := W_{\alpha\mu\nu\beta} n^\mu n^\nu, \quad B_{\alpha\beta} := W^*_{\alpha\mu\nu\beta} n^\mu n^\nu. \quad (7)$$

Theorem 2 *Let Σ_t be a $t = \text{const.}$ Cauchy surface in the Minkowski space-time. Charge $C(W, Q)$, defined by (5), can be decomposed at each Σ_t surface as follows:*

$$C(W, Q) = \alpha(t)I(E, X) + \beta(t)I(B, Y), \quad (9)$$

where $I(E, X)$ is an instant charge (8); $\alpha(t)$, $\beta(t)$ are quadratic polynomials of a single indeterminate t .

Conformally flat three-dimensional space contains ten-dimensional space of CKV. We discuss below the physical properties of some of them.

Mass is one of the most important quantities which characterize a physical system. However, some subtleties arise and they will be analyzed in our next paper [4]. We claim that the following instant charge:

$$I_{\mathcal{M}} := I(E, \mathcal{S}) \quad (10)$$

describes quasi-local mass. $\mathcal{S}$ is CKV related with scaling generator⁶. If we consider asymptotically flat space with two-dimensional foliation of topological spheres $S(R)$, which are parametrized by their radius R , then the ADM mass is given by $m_{\text{ADM}} = \frac{1}{8\pi} \lim_{R \rightarrow \infty} I_{\mathcal{M}}(S(R))$. The theorems below illustrate the relation between instant charges and traditional Poincaré (ADM) charges for momentum and angular momentum:

Theorem 3 *Let Σ be a flat, three-dimensional spatial hypersurface immersed in the spacetime which satisfies Einstein vacuum equations. Assuming that $B^i_{j|i} = 0$, then for any two-dimensional, closed surface A immersed in Σ holds*

$$\int_A B^i_j \mathcal{R}^j_k dS_i = \int_A P^i_j \mathcal{T}^j_k dS_i, \quad (11)$$

where P^i_j is (canonical) ADM momentum, $\mathcal{R}_k$ is a rotation generator around the axis k , and $\mathcal{T}_k$ is a translation generator along the axis k .

Theorem 4 *Let Σ be a flat, three-dimensional spatial hypersurface immersed in the spacetime which satisfies Einstein vacuum equations. Assuming that the linear momenta vanish*

$$I(B, \mathcal{T}_x) = I(B, \mathcal{T}_y) = I(B, \mathcal{T}_z) = 0 \quad (12)$$

and the three-dimensional covariant divergence of the magnetic part disappears

$$B^i_{j|i} = 0, \quad (13)$$

⁶ For conformally flat space, with the metric in the form $\psi (dR^2 + R^2 d\Omega^2)$, the scaling generator has a form of $\mathcal{S} = R\partial_R$.

then for any two-dimensional, closed surface A immersed in Σ holds

$$\int_A B^i_j \mathcal{K}_k^j dS_i = \int_A P^i_j \mathcal{R}_k^j dS_i, \quad (14)$$

where P^i_j is the ADM momentum, $\mathcal{K}_k$ is the generator of conformal acceleration in the direction of k , and $\mathcal{R}_k$ is a rotation generator around the axis k .

2.1. Example: Constant-time surface in Schwarzschild–de Sitter spacetime

Let us consider Schwarzschild–de Sitter spacetime in the standard coordinates (t, r, θ, ϕ) , whose line element is given by the formula

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \quad (15)$$

where $f(r) = 1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2$. Three-dimensional surface $\Sigma_t = \{t = \text{const.}\}$ is conformally flat $\frac{dr^2}{f(r)} + r^2 d\Omega^2 = \frac{r^2}{R^2} [dR^2 + R^2 d\Omega]$. The above equation determines R up to multiplicative constant. However, the three-dimensional CKV related with scaling generator is uniquely determined. It is equal to $\mathcal{S} = R\partial_R$. Projection of four-dimensional Weyl tensor with a well-defined $\mathcal{S}$ enables one to construct quasi-local mass (10). For the considered case, the mass does not depend on the choice of two-dimensional surface⁷ and it is constant

$$I_{\mathcal{M}} = 8\pi M. \quad (16)$$

2.2. Asymptotic charges

Instant charges, defined by (8), do not depend on the two-dimensional integration surface iff the three-dimensional divergence of the integrand vanishes. Let us weaken the assumptions for instant charges by considering a three-dimensional spatial surface Σ equipped with the Riemannian metric γ_{ij} . We consider two two-dimensional, oriented closed surfaces A_1, A_2 limiting the three-dimensional volume $V \subset \Sigma$ such that the boundary of V fulfills $\partial V = A_1 \cup A_2$. At the moment, we do not impose additional conditions for gravito-electro-magnetic tensor E_{ij} and vector field X^k . Using the Stokes theorem, we have

$$\begin{aligned} \int_{\partial V} (\sqrt{\gamma} E^{ij} X_j) dA_i &= \int_V (\sqrt{\gamma} E^{ij} X_j)_{;i} d\Sigma \\ &= \int_V \left[\sqrt{\gamma} E^{ij} \left(X_{(i|j)} - \frac{1}{3} X^k{}_{|k} \gamma_{ij} \right) + \sqrt{\gamma} X_j E^{ij}{}_{|i} \right] d\Sigma, \end{aligned} \quad (17)$$

⁷ The two-dimensional surface has to be homotopic to a round sphere — an orbit of rotational symmetry of the Schwarzschild–de Sitter solution.

where “|” denotes a three-dimensional Levi-Civita derivative which agreed with the metric γ_{ij} .

If space Σ possesses a solution of CKV asymptotically⁸ and the divergence of gravito-electric tensor also vanishes at spatial infinity, then the following asymptotic charge:

$$I_{\text{as}}(E, X) := \lim_{r \rightarrow \infty} \int_{S(r)} E^i_j X^j dS_i \quad (18)$$

stabilizes at spatial infinity providing integral (17) converges at spatial infinity. It means that the integrand should fall off faster than $\frac{1}{r}$. The precise analysis will be discussed in [4].

3. Conclusions

The construction presented in the paper gives a clear relation between instant charges (5) and conserved quantities built with the help of CYK tensor. Such a relation may be specially useful in the case of asymptotically de Sitter spacetimes. There exist significant interpretation problems for asymptotically de Sitter spacetimes, mainly related with the spatial nature of the *scri* beyond the cosmological horizon. Mass (energy) defined as an instant charge does not contain a CKV which changes its spacetime characteristics when passing the cosmological horizon. In some sense, the construction briefly described in the paper is an introduction to further investigations for de Sitter spacetime. All CYK solutions for de Sitter spacetime have been analyzed in [1]. The CYK decomposition for de Sitter spacetime will be a part of a separate paper.

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⁸ By an asymptotic solution of CKV we mean there exists a vector field X^k which satisfies $X_{i|j} - \lambda g_{ij} = O\left(\frac{1}{r^\delta}\right)$, where δ is suitably chosen to the case.