FIERZ THEORY VERSUS LINEAR GRAVITY*

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One of the most popular points of view on linearized gravity is massless spin-2 particle theory. This theory is often used as a starting point to formulate a quantum version of gravity theory. The spin-2 field has the well-defined local density of energy equal to $\frac{1}{2}(E^2 + B^2)$ in analogy to Maxwell electrodynamics. However, energy in linearized gravity is non-local. The relations and differences between linearized gravity and the spin-2 field theory will be discussed in this paper.

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1. Introduction

Study of spin-2 field theory in linearized gravity context is motivated by the fact that fulfilling the linearized Einstein Equations by some linearized metric (symmetric field on Minkowski background) is equivalent to fulfilling Bianchi identities by the linearized Weyl tensor of this metric. The Weyl tensor for the Levi-Civita connection can be identified with spin-2 field because of its algebraic symmetries. This way, spin-2 field theory can become a different picture of linearized gravity. However, the gauge-independent formula for the amount of energy contained in a bounded region V is local for every tensor field theory on flat background. For linearized gravity, the situation is different because gauge transformations represent infitesimal diffeomorphisms of spacetime. In consequence, gauge-independent expression of energy cannot be local. To clarify differences between linearized gravity and spin-2 field theory, at first, let us focus on connections between these two theories.

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2. From linearized gravity to spin-2 field

Definition 1. *The following properties:*

$$W_{\mu\alpha\nu\beta} = W_{\nu\beta\mu\alpha} = W_{[\mu\alpha][\nu\beta]}\,, \qquad W_{\mu[\alpha\nu\beta]} = 0\,, \qquad \eta^{\mu\nu} W_{\mu\alpha\nu\beta} = 0 \tag{1}$$

can be used as a definition of spin-2 field W.

Due to the way of constructing theories for fields of a given spin [1], equations

$$\nabla^{\alpha} W_{\alpha\beta\mu\nu} = 0 \tag{2}$$

are massless spin-2 field equations. Simple derivation shows that these equations are equivalent to Bianchi equations

$$\nabla_{[\lambda} W_{\mu\nu]\alpha\beta} = 0\,. \tag{3}$$

However, the well-known decomposition of the Riemann tensor

$$\begin{aligned} R_{\alpha\beta\mu\nu} &= W_{\alpha\beta\mu\nu} + \tfrac{1}{2}\left(R_{\alpha\mu}g_{\beta\nu} - R_{\alpha\nu}g_{\beta\mu} + R_{\beta\nu}g_{\alpha\mu} - R_{\beta\mu}g_{\alpha\nu}\right) \\ &\quad + \tfrac{1}{6} R\left(g_{\alpha\mu}g_{\beta\nu} - g_{\alpha\nu}g_{\beta\mu}\right) \end{aligned}$$

(where $W_{\alpha\beta\mu\nu}$ is the Weyl tensor, $R_{\alpha\mu}$ is the Ricci tensor and R is the Ricci scalar) shows that vacuum Einstein Equations imply Bianchi identities for the Weyl tensor

$$R_{\alpha\mu} = 0 \quad \Rightarrow \quad \nabla_{[\lambda} W_{\mu\nu]\alpha\beta} = 0\,.$$

Finally, from Definition 1, we obtain that the Weyl tensor for metric connection can be interpreted as a spin-2 field, and equations of linearized gravity can be interpreted as equations of some spin-2 field theory on Minkowski background.

If we perform 3+1 decomposition of spacetime, the spin-2 field W decomposes into traceless, symmetric tensors E_{kl} and B_{kl}

$$W_{0k0l} =: E_{kl}\,, \qquad \epsilon_{k}{}^{ij} W^{0}{}_{lij} =: B_{kl}\,. \tag{4}$$

Equations (3) expressed by these tensors, have a form of

$$\mathrm{div}E = 0\,, \qquad \mathrm{curl}E = -\dot{B}\,, \tag{5}$$

$$\mathrm{div}B = 0\,, \qquad \mathrm{curl}B = \dot{E}\,, \tag{6}$$

where $(\mathrm{div}A)^l := A^{kl}{}_{;k}$ and $(\mathrm{curl}A)_{kl} := \epsilon_{(k}{}^{ij} A_{l)i;j}$. This form shows analogy between spin-2 field theory and electrodynamics, which motivates us to consider a similar theory, where spin-2 field comes from derivation of some potential A.

3. Fierz–Lanczos theory

Due to [2], the spin-2 field $W_{\alpha\beta\mu\nu}$ is built from the first derivatives of tensor $A_{\alpha\beta\gamma}$ which fulfills algebraic conditions

$$A_{\alpha\beta\gamma} = -A_{\beta\alpha\gamma}\,, \qquad A_{[\alpha\beta\gamma]} = 0\,, \qquad A_{\alpha\mu}{}^{\mu} = 0 \tag{7}$$

with the following formula which expresses W in terms of A:

$$\begin{aligned} W_{\alpha\beta\mu\nu} :\ =\ & r_{\alpha\beta\mu\nu} - \tfrac{1}{2}\left(r_{\alpha\mu}\eta_{\beta\nu} - r_{\alpha\nu}\eta_{\beta\mu} + \eta_{\alpha\mu}r_{\beta\nu} - \eta_{\alpha\nu}r_{\beta\mu}\right) \\ & + \tfrac{1}{6}\left(\eta_{\alpha\mu}\eta_{\beta\nu} - \eta_{\alpha\nu}\eta_{\beta\mu}\right) r\,, \end{aligned} \tag{8}$$

where $r_{\alpha\beta\mu\nu} := A_{\alpha\beta\nu;\mu} - A_{\alpha\beta\mu;\nu} + A_{\nu\mu\alpha;\beta} - A_{\nu\mu\beta;\alpha}$, $r_{\alpha\beta} = r^{\mu}{}_{\alpha\mu\beta}$, $r = r_{\mu\nu}\eta^{\mu\nu}$. Potential A in 3+1 splitting, decomposes into symmetric, traceless tensors P and S, and two covectors a and b

$$p_{kl} = -A_{0(kl)}\,, \qquad a_i = -A_{0i0}\,, \tag{9}$$

$$S_{kl} = -\tfrac{1}{2}A_{ij(k}\epsilon^{ij}{}_{l)}\,, \qquad b_i = -\tfrac{1}{2}\epsilon_i{}^{kl}A_{kl0}\,. \tag{10}$$

E and B in terms of the above quantities express as follows:

$$E = -\dot p + \mathrm{curl}\ S + \tfrac{3}{2}TS(\nabla a)\,, \qquad B = \dot S + \mathrm{curl}\ p - \tfrac{3}{2}TS(\nabla b)\,, \tag{11}$$

where $TS(t)$ denotes traceless and symmetric part of tensor t. Equations (1) can be obtained from the variational formula for the Lagrangian density (see [3])

$$\begin{aligned} L\ &=\ \tfrac{1}{16}\sqrt{|\det g|}\,w_{\lambda\mu\nu\kappa}w^{\lambda\mu\nu\kappa} = \tfrac{1}{2}\sqrt{|\det g|}\left(E^2 - B^2\right) \\ &=\ \tfrac{1}{2}\left\{\left(\dot p - \mathrm{curl}\ S - \tfrac{3}{2}TS(\nabla a)\right)^2 - \left(\dot S + \mathrm{curl}\ p - \tfrac{3}{2}TS(\nabla b)\right)^2\right\}. \end{aligned} \tag{12}$$

Variation of Lagrangian for fields fulfilling field equations is given by

$$\begin{aligned} \delta L\ &=\ \int_V \left\{D\delta\left(\dot p - \mathrm{curl}\, S - \tfrac{3}{2}TS(\nabla a)\right) - H\delta\left(\dot S + \mathrm{curl}\, p - \tfrac{3}{2}TS\nabla b\right)\right\} \\ &=\ \int_V \left\{-D\delta\dot p - \mathrm{curl}\, D\delta S - H\delta\dot S + \mathrm{curl}\, H\delta p\right\} \\ &=\ -\int_V \left(D\delta\dot p + \dot D\delta p + H\delta\dot S + \dot H\delta S\right)\,, \end{aligned} \tag{13}$$

where D and H are momenta canonically conjugate to p and S

$$-D = \frac{\partial L}{\partial \dot p} = -E\,, \qquad -H = \frac{\partial L}{\partial \dot S} = -B\,. \tag{14}$$

Spin-2 equations expressed by potentials (p, S, a, b) have a form of

$$\tfrac{3}{2}TS\left(\nabla\left(\dot{a}+\tfrac{1}{2}\mathrm{curl}\, b\right)\right) = \ddot{p}+\mathrm{curl}\,\mathrm{curl}\, p\,, \quad (15)$$

$$\tfrac{3}{2}TS\left(\nabla\left(\dot{b}-\tfrac{1}{2}\mathrm{curl}\, a\right)\right) = \ddot{S}+\mathrm{curl}\,\mathrm{curl}\, S\,. \quad (16)$$

This is the Fierz–Lanczos formulation of spin-2 field theory, and now let us use it to derive formula for spin-2 field energy contained in a given bounded region V.

4. Energy in Fierz–Lanczos theory

If we establish 3+1 decomposition of spacetime, we can consider on space of spin-2 fields on a bounded region V with proper boundary conditions, such that we can integrate by parts with vanishing boundary terms. Every three-dimensional symmetric and traceless tensor t decomposes into three parts (see [4])

$$t_{kl} = t^{\mathrm{T}}_{kl} + t^{\mathrm{V}}_{kl} + t^{\mathrm{S}}_{kl}\,,$$

where

$$\mathrm{div}\, t^{\mathrm{T}} = 0 \ \mathrm{Tr}\left(t^{\mathrm{T}}\right) = 0\,,\ t^{\mathrm{V}}_{kl} = TS(\nabla\xi)_{kl}\ \mathrm{div}\,\xi = 0\,,\ t^{\mathrm{S}}_{kl} = f_{,kl} - \tfrac{1}{3}\delta_{kl} f^{,i}{}_{i} \quad (17)$$

for some function f and covector ξ. The t^{T} part of the tensor t we will call *traceless-transversal* part of t. From the boundary conditions, we obtain that in our region V these three components are orthogonal with respect to the scalar product $(A|B) := \int_{\mathrm{V}} A^{ij}B_{ij}\mathrm{d}^3x$. Now let us consider the fields fulfilling the spin-2 equations. From (5), (6) and (14), we have $E = E^{\mathrm{T}} = D^{\mathrm{T}} = D$ and $B = B^{\mathrm{T}} = H^{\mathrm{T}} = H$. *Traceless-transversal* parts of equations (15), (16) are

$$\Box p = \ddot{p} - \Delta p = \ddot{p} + \mathrm{curl}\,\mathrm{curl}\, p = 0\,, \qquad \Box S = \ddot{S} - \Delta S = \ddot{S} + \mathrm{curl}\,\mathrm{curl}\, S = 0\,. \quad (18)$$

On space of *traceless-transversal* tensors, the curl operator is invertible, so we can write $S^{\mathrm{T}} = \mathrm{curl}^{-1} h^{\mathrm{T}}$, and from (18) we have $\ddot{h}^{\mathrm{T}} + \mathrm{curl}\,\mathrm{curl}\, h^{\mathrm{T}} = 0$.

Having this, we can reduce symplectic space of configurations (p, S, D, H) with symplectic form $\omega = \delta D \wedge \delta p + \delta H \wedge \delta S$ onto $(D, \widetilde{p})$ with symplectic form $\omega = \delta D \wedge \delta\widetilde{p}$

$$-\delta L = \int_{\mathrm{V}}\left(D\delta\dot{p} + \dot{D}\delta p + H\delta\dot{S} + \dot{H}\delta S\right) = \int_{\mathrm{V}}\left(D\delta\dot{p}^{\mathrm{T}} + \dot{D}\delta p^{\mathrm{T}} + H\delta\dot{S}^{\mathrm{T}} + \dot{H}\delta S^{\mathrm{T}}\right)$$

$$= \int_{\mathrm{V}}\left(D\delta\dot{p}^{\mathrm{T}} + \dot{D}\delta p^{\mathrm{T}} + \mathrm{curl}\, H\delta\dot{h}^{\mathrm{T}} - D\delta\mathrm{curl}\, S^{\mathrm{T}}\right)$$

$$
\begin{aligned}
&= \int\limits_{\mathrm{V}} \left(D\delta\dot{p}^{\mathrm{T}} + \dot{D}\delta p^{\mathrm{T}} + \dot{D}\delta\dot{h}^{\mathrm{T}} - D\delta \operatorname{curl}\operatorname{curl} h \right) \\
&= \int\limits_{\mathrm{V}} \left(D\delta\left(\dot{p}^{\mathrm{T}} + \ddot{h}^{\mathrm{T}}\right) + \dot{D}\delta\left(p^{\mathrm{T}} + \dot{h}^{\mathrm{T}}\right)\right) = \int\limits_{\mathrm{V}} \left(D\delta\dot{\widetilde{p}} + \dot{D}\delta\widetilde{p} \right) , \qquad (19)
\end{aligned}
$$

where we denoted $\widetilde{p} := p^{\mathrm{T}} + \dot{h}^{\mathrm{T}}$. Now we can derive (an energy form) Hamiltonian H performing a standard Legendre transformation

$$
\begin{aligned}
-H &= L + D\dot{\widetilde{p}} = \tfrac{1}{2}\left(E^2 - B^2\right) + E\left(\dot{p}^{\mathrm{T}} + \ \ddot{h}^{\mathrm{T}}\right) \\
&= \tfrac{1}{2}\left(E^2 - B^2\right) + E\left(\dot{p}^{\mathrm{T}} - \operatorname{curl}\operatorname{curl} h\right) \\
&= \tfrac{1}{2}\left(E^2 - B^2\right) - E\left(-\dot{p}^{\mathrm{T}} + \operatorname{curl} S^{\mathrm{T}}\right) \\
&= \tfrac{1}{2}\left(E^2 - B^2\right) - E \cdot E^{\mathrm{T}} = -\tfrac{1}{2}\left(E^2 + B^2\right) .
\end{aligned}
$$

5. Summary

The calculus in the last section showed that energy in spin-2 field theory is local. It is the main difference between spin-2 field theory and linearized gravity. Both theories have the same equations, but the energy is different. For spin-2 field theory, it is equal $\frac{1}{2}\int_V (E^2 + B^2)$, but in linearized gravity energy is given by $\frac{1}{2}\int_V (E\varDelta^{-1}E + B\varDelta^{-1}B)$ (see [5]), hence it is non-local.

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