ON THE RELATION BETWEEN THE CANONICAL HAMILTON–JACOBI EQUATION AND THE DE DONDER–WEYL HAMILTON–JACOBI FORMULATION IN GENERAL RELATIVITY*

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A relation between the canonical Hamilton–Jacobi (HJ) theory and the De Donder–Weyl Hamilton–Jacobi (DWHJ) theory in the calculus of variations is studied. In the case of a scalar field in curved space-time and in general relativity in Gaussian coordinates, we show how the functional derivative canonical HJ equation is derived from the partial derivative DWHJ equation. The derivation is based on the split between space and time and the Ansatz which relates the HJ functional eikonal on the infinite dimensional space of initial data with the DWHJ eikonal functions on the finite dimensional space of field variables and space-time coordinates.

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1. Introduction

The De Donder–Weyl (DW) theory [1–5] is a generalization of the Hamiltonian formulation to field theory which does not distinguish between space and time. Given a Lagrangian $L(y^a, y^a_\mu, x^\nu)$ depending on the space-time variables x^ν, field variables y^a, and their first jet coordinates y^a_μ (such that a restriction to a field configuration $y^a = y^a(x^\nu)$ implies: $y^a_\mu = \partial y^a(x)/\partial x^\mu =:$

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$\partial_\mu y^a$), a Legendre transformation to new variables $p^\mu_a := \frac{\partial L}{\partial y^a_\mu}$ (polymomenta) and $H := y^a_\mu p^\mu_a - L$ (the DW Hamiltonian) enables us to write the Euler–Lagrange field equations in the DW Hamiltonian form

$$\partial_\mu y^a = \frac{\partial H}{\partial p^\mu_a}\,, \qquad \partial_\mu p^\mu_a = -\frac{\partial H}{\partial y^a}\,, \tag{1}$$

provided the regularity condition $\det\left(\frac{\partial^2 L}{\partial y^a_\mu \partial y^b_\nu}\right) \neq 0$ is fulfilled.

The dynamical content of the DW Hamiltonian formulation can be also represented by the DW Hamilton–Jacobi (DWHJ) equation [1, 2, 4]

$$\partial_\mu S^\mu + H\left(y^a, \frac{\partial S^\mu}{\partial y^a}, x^\mu\right) = 0\,. \tag{2}$$

This partial differential equation for the eikonal functions $S^\mu(y^a, x^\mu)$ determines the solutions of (1) by the embedding conditions

$$\frac{\partial L}{\partial y^a_\mu} = \frac{\partial S^\mu}{\partial y^a}\,. \tag{3}$$

Geometrical aspects of the DWHJ equation have been recently studied in [6, 7]. The historical role of the HJ formulation of mechanics in the discovery of the Schrödinger equation [8] makes the DWHJ formulation particularly interesting. In fact, within the framework of precanonical quantization which uses the DW theory instead of the canonical Hamiltonian formalism [9, 10], it was already shown that the DWHJ equation for scalar fields follows from the corresponding precanonical generalization of the Schrödinger equation in the classical limit [11]. Precanonical quantization has been applied to the quantum Yang–Mills theory [12–14], quantum scalar field theory in curved spacetime [15–18], and to quantization of gravity in metric [19, 20] and vielbein variables [21–25].

To understand the connection between this approach and the canonical quantization of general relativity [26, 27], we investigate here a relationship between the DWHJ equation for general relativity [1, 28, 29] and the canonical HJ equation [30] that has been used to explore the semiclassical approximation of canonical quantum gravity [31–33]. In Section 2, we establish a relation between the DWHJ equation and the canonical HJ equation for a scalar field in a general curved space-time. In Section 3, we restrict ourselves to Gaussian coordinates and derive the canonical HJ equation for general relativity from the DWHJ formulation. Our results generalize the relation between the DWHJ and the canonical HJ equation in flat space-time found by Kanatchikov [34] and applied to the bosonic string by Nikolić [35].

2. Canonical HJ vs. DWHJ for the scalar field in curved space-time

The Lagrangian density of a scalar field in curved spacetime with the metric $g_{\mu\nu}$ ($g := \det g_{\mu\nu}$) reads

$$L = -\frac{1}{2} g^{\mu\nu} \varphi_\mu \varphi_\nu \sqrt{-g} - V(\varphi)\sqrt{-g}\,. \tag{4}$$

The canonical HJ equation is derived by using a foliation of spacetime by space-like hypersurfaces $\mathcal{F}$ labelled by the time function t.

In adapted coordinates, the metric $g_{\mu\nu}$ decomposes in terms of the lapse function N, the shift vector N^i and the spatial components h_{ij}

$$\mathrm{d}s^2 = \left(N_i N^i - N^2\right) \mathrm{d}t^2 + 2N_i \mathrm{d}x^i \mathrm{d}t + h_{ij} \mathrm{d}x^i \mathrm{d}x^j\,. \tag{5}$$

By introducing the canonical momentum π and the canonical Hamiltonian density $\mathcal{H}(\phi(\boldsymbol{x}), \pi(\boldsymbol{x}))$ in the usual way (*cf. e.g.* [36]), the canonical HJ equation for the eikonal functional $\boldsymbol{S}(\varphi(\boldsymbol{x}), \boldsymbol{x}, t)$ takes the explicit form ($h = \det\, h_{ij}$)

$$\begin{aligned}&\partial_t \boldsymbol{S}\\ &+\int \mathrm{d}\boldsymbol{x} \left(\frac{N}{2\sqrt{h}} \frac{\delta \boldsymbol{S}}{\delta\varphi(\boldsymbol{x})} \frac{\delta \boldsymbol{S}}{\delta\varphi(\boldsymbol{x})} + \frac{1}{2} h^{ij} \partial_i\varphi \partial_j\varphi \sqrt{-g} + V(\varphi)\sqrt{-g} + N^i \partial_i\varphi \frac{\delta \boldsymbol{S}}{\delta\varphi(\boldsymbol{x})} \right)\\ &= 0\,,\end{aligned} \tag{6}$$

where the solutions of field equations are embedded by the condition

$$\sqrt{-g} g^{0\mu} \partial_\mu \varphi = -\frac{\delta \boldsymbol{S}}{\delta\varphi(\boldsymbol{x})}\,.$$

On the other hand, from (4) we obtain the polymomenta and the DW Hamiltonian density

$$p^\mu = \frac{\partial L}{\partial \varphi_\mu} = -\sqrt{-g}\, \varphi_\nu\, g^{\mu\nu}\,, \qquad H = -\frac{1}{2\sqrt{-g}} p^\mu p^\nu g_{\mu\nu} + V(\varphi)\sqrt{-g}\,, \tag{7}$$

and the DWHJ equation (2) for the eikonal densities $S^\mu(\varphi, x^\nu)$

$$\partial_\mu S^\mu - \frac{1}{2\sqrt{-g}} \frac{\partial S^\mu}{\partial \varphi} \frac{\partial S^\nu}{\partial \varphi} g_{\mu\nu} + V(\varphi)\sqrt{-g} = 0\,, \tag{8}$$

where $p^\mu = \frac{\partial S^\mu}{\partial \phi}$ and the solutions of classical field equations are given by the embedding condition

$$\frac{\partial S^\mu}{\partial \varphi} = -\sqrt{-g} g^{\mu\nu} \partial_\nu \varphi\,. \tag{9}$$

Our task is to understand the relationship between the canonical formulation (6), which requires a space-time split, and the DW formulation (8) where all space-time variables are treated equally. Here, we generalize the consideration of [34] to curved spacetime. At first, we introduce the restriction of the densities $S^\mu(\varphi, x^\nu)$ to a field configuration $\varphi(\boldsymbol{x})$ on a hypersurface $\mathcal{F}$ at a time t, $S^\mu|_{\varphi(\mathcal{F})} := S^\mu(\varphi(\boldsymbol{x}), \boldsymbol{x}, t)$. The embedding condition (9) in adapted coordinates yields

$$
\begin{aligned}
\left.\frac{\partial S^i}{\partial \varphi}\right|_{\varphi(\mathcal{F})} + N^i \left.\frac{\partial S^0}{\partial \varphi}\right|_{\varphi(\mathcal{F})} &= -\sqrt{-g}\,\partial_j\varphi\, h^{ij}\,, && (10)\\
\left.\frac{\partial S^0}{\partial \varphi}\right|_{\varphi(\mathcal{F})} N^2 &= \sqrt{-g}\left(\partial_0\varphi - N^i\partial_i\varphi\right)\,. && (11)
\end{aligned}
$$

Following [34], we construct the eikonal functional from the DW eikonal densities

$$
\boldsymbol{S} := \int\limits_{\mathcal{F}} \mathrm{d}\boldsymbol{x}\, S^0\big|_{\varphi(\mathcal{F})}\,. \tag{12}
$$

Then, using (8) and (10), for the time derivative of $\boldsymbol{S}$, we obtain

$$
\begin{aligned}
\partial_t \boldsymbol{S} &= \int \mathrm{d}\boldsymbol{x}\, \partial_t S^0(\varphi(\boldsymbol{x}), \boldsymbol{x}, t) = \int \mathrm{d}\boldsymbol{x} \Bigg\{ -\frac{\mathrm{d}}{\mathrm{d}x^i} S^i\Big|_{\varphi(\mathcal{F})} - \frac{N}{2\sqrt{h}} \left(\left.\frac{\partial S^0}{\partial\varphi}\right|_{\varphi(\mathcal{F})}\right)^2 \\
&\quad -N^i\partial_i\varphi\, \left.\frac{\partial S^0}{\partial\varphi}\right|_{\varphi(\mathcal{F})} - \frac{1}{2}\sqrt{-g}h^{ij}\partial_i\varphi\partial_j\varphi - \sqrt{-g}\,V(\varphi(\boldsymbol{x}))\Bigg\}\,,
\end{aligned} \tag{13}
$$

where the notation for the total divergence of the eikonal density on $\varphi(\mathcal{F})$ is introduced

$$
\frac{\mathrm{d}}{\mathrm{d}x^i} S^i|_{\varphi(\mathcal{F})} = \left.\frac{\partial S^i}{\partial\varphi}\right|_{\varphi(\mathcal{F})} \frac{\partial\varphi(\boldsymbol{x})}{\partial x^i} + \left.\frac{\partial S^i}{\partial x^i}\right|_{\varphi(\mathcal{F})}\,. \tag{14}
$$

By noticing that the functional derivative of the functional (12) with respect to $\varphi(\boldsymbol{x})$ reads

$$
\frac{\delta \boldsymbol{S}}{\delta\varphi(\boldsymbol{x})} = \left.\frac{\partial S^0}{\partial\varphi}\right|_{\varphi(\mathcal{F})}\,, \tag{15}
$$

and assuming that $S^i\,|_{\varphi(\mathcal{F})}$ vanishes at the boundary of $\varphi(\mathcal{F})$, so that the integral of the total divergence in (13) does not contribute, we conclude that the functional $\boldsymbol{S}$ constructed in (12) obeys the canonical HJ equation (6) as a consequence of the DWHJ equation (8) for the eikonal densities S^μ.

3. The DWHJ equation and the canonical HJ equation in general relativity

In $3+1$ dimensions, the DWHJ equation for general relativity found by De Donder [1] and Hořava [28, 29] reads

$$\partial_\mu S^\mu + \mathfrak{g}^{\alpha\gamma}\left(\frac{\partial S^\delta}{\partial \mathfrak{g}^{\alpha\beta}}\frac{\partial S^\beta}{\partial \mathfrak{g}^{\gamma\delta}} - \frac{1}{3}\frac{\partial S^\beta}{\partial \mathfrak{g}^{\alpha\beta}}\frac{\partial S^\delta}{\partial \mathfrak{g}^{\gamma\delta}}\right) = 0\,. \tag{16}$$

It uses the metric density components $\mathfrak{g}^{\alpha\beta} = \sqrt{-g}\, g^{\alpha\beta}$ as the field variables, so that $S^\mu = S^\mu(\mathfrak{g}^{\alpha\beta}, x^\nu)$.

The polymomenta derived from the truncated Hilbert action without the surface term are expressed in terms of the Christoffel symbols

$$Q^\alpha_{\beta\gamma} = \frac{1}{2}\left(\delta^\alpha_\beta \Gamma^\mu_{\gamma\mu} + \delta^\alpha_\gamma \Gamma^\mu_{\beta\mu}\right) - \Gamma^\alpha_{\beta\gamma}\,. \tag{17}$$

The solutions of Einstein's field equations are constructed from the eikonal densities S^μ using the embedding condition

$$Q^\alpha_{\beta\gamma} = \frac{\partial S^\alpha}{\partial \mathfrak{g}^{\beta\gamma}} = \frac{1}{2}\left(\delta^\alpha_\beta \Gamma^\mu_{\gamma\mu} + \delta^\alpha_\gamma \Gamma^\mu_{\beta\mu}\right) - \Gamma^\alpha_{\beta\gamma} \tag{18}$$

and the well-known expression of the Christoffel symbols in terms of the first derivatives of the metric.

In order to understand the relation between the DWHJ equation for general relativity (16) and the canonical HJ equation found by Peres [30]

$$\int \mathrm{d}\boldsymbol{x}\left(\sqrt{-g}\;{}^3R + \frac{1}{\sqrt{-g}}\left(\frac{1}{2}g_{ij}g_{kl} - g_{ik}g_{il}\right)\frac{\delta \boldsymbol{S}}{\delta g_{ij}}\frac{\delta \boldsymbol{S}}{\delta g_{kl}}\right) = 0\,, \tag{19}$$

we have to perform a space-time decomposition in (16). In this paper, we confine ourselves to the simpler case of adapted (Gaussian) coordinates with $g_{0i} = N_i = 0$ and $g_{00} = -1$. Following [34], we construct the canonical HJ functional $\boldsymbol{S}([g_{ij}(\boldsymbol{x})], t)$ from the eikonal densities $S^\mu(\mathfrak{g}^{\alpha\beta}, x^\mu)$

$$\boldsymbol{S}\left([g_{ij}(\boldsymbol{x})], t\right) := \int_{\mathcal{F}} \mathrm{d}\boldsymbol{x}\; S^0\big|_{\mathfrak{g}(\mathcal{F})} = \int_{\mathcal{F}} \mathrm{d}\boldsymbol{x}\, S^0\left(\mathfrak{g}^{\alpha\beta}(\boldsymbol{x}), \boldsymbol{x}, t\right)\,, \tag{20}$$

where $S^\mu|_{\mathfrak{g}(\mathcal{F})} \equiv S^\mu(\mathfrak{g}^{\alpha\beta}(\boldsymbol{x}), \boldsymbol{x}, t)$ denotes the restriction of the eikonal densities $S^\mu\left(\mathfrak{g}^{\beta\gamma}, x^\mu\right)$ to the spatial field configurations $\mathfrak{g}^{\alpha\beta}(\boldsymbol{x})$ on a hypersurface $\mathcal{F}$ at the time t. In the Gaussian coordinates, the embedding conditions (18) give rise to the following relations:

$$\frac{\partial S^i}{\partial \mathfrak{g}^{0j}} + \frac{1}{2}\delta^i_j \frac{\partial S^0}{\partial \mathfrak{g}^{kl}} g^{kl} - \frac{\partial S^0}{\partial \mathfrak{g}^{jk}} g^{ki} = 0\,, \qquad \frac{\partial S^i}{\partial \mathfrak{g}^{00}} = 0\,, \qquad \frac{\partial S^0}{\partial \mathfrak{g}^{00}} + \frac{\partial S^0}{\partial \mathfrak{g}^{ij}} g^{ij} = 0\,, \tag{21a}$$

$$\frac{\partial S^i}{\partial \mathfrak{g}^{jk}} = \frac{1}{2}\left(\delta^i_j \Gamma^l_{kl} + \delta^i_k \Gamma^l_{jl}\right) - \Gamma^i_{jk}\,, \qquad \frac{\partial S^0}{\partial \mathfrak{g}^{0i}} = \frac{1}{2}\Gamma^j_{ij}\,, \tag{21b}$$

$$\frac{\partial S^0}{\partial \mathfrak{g}^{ij}} = -\Gamma^0_{ij} = -\frac{1}{2}\partial_0 g_{ij}\,. \tag{21c}$$

Using (16) and (21a), (21b), we obtain for the time derivative of $\boldsymbol{S}$

$$\begin{aligned}\partial_t \boldsymbol{S} \;=& \int \mathrm{d}\boldsymbol{x} \left\{ -\frac{\mathrm{d}}{\mathrm{d}x^i} S^i \bigg|_{\mathfrak{g}(\mathcal{F})} + \frac{\partial S^i}{\partial \mathfrak{g}^{\alpha\beta}}\bigg|_{\mathfrak{g}(\mathcal{F})} \partial_i \mathfrak{g}^{\alpha\beta} + \mathfrak{g}^{ij}\left(\Gamma^l_{kl}\Gamma^k_{ij} - \Gamma^k_{il}\Gamma^l_{jk}\right)\right. \\ & \left. + \frac{1}{\sqrt{-g}}\left(\frac{\partial S^0}{\partial \mathfrak{g}^{ij}}\bigg|_{\mathfrak{g}(\mathcal{F})} \mathfrak{g}^{ij}\right)^2 - \frac{1}{\sqrt{-g}}\frac{\partial S^0}{\partial \mathfrak{g}^{ij}}\bigg|_{\mathfrak{g}(\mathcal{F})} \frac{\partial S^0}{\partial \mathfrak{g}^{kl}}\bigg|_{\mathfrak{g}(\mathcal{F})} \mathfrak{g}^{ik}\mathfrak{g}^{jl} \right\}, \end{aligned} \tag{22}$$

where the total divergence is understood as in (14). Then, by using (21b), and the identity

$$\frac{\partial}{\partial g_{\alpha\beta}} = \sqrt{-g}\left(-g^{\alpha\mu}g^{\beta\nu} + \frac{1}{2}g^{\mu\nu}g^{\alpha\beta}\right)\frac{\partial}{\partial \mathfrak{g}^{\mu\nu}}\,, \tag{23}$$

we find

$$\begin{aligned} & \frac{\partial S^i}{\partial \mathfrak{g}^{\alpha\beta}}\bigg|_{\mathfrak{g}(\mathcal{F})} \partial_i \mathfrak{g}^{\alpha\beta} \; + \sqrt{-g}\left(\Gamma^m_{km}\Gamma^k_{lc}\,g^{lc} - \Gamma^k_{ij}\Gamma^j_{lk}g^{li}\right) \\ &= \sqrt{-g}\;{}^3R + \partial_i\left(\sqrt{-g}\left(g^{mi}\Gamma^k_{mk} - g^{ke}\Gamma^i_{ke}\right)\right)\,. \end{aligned} \tag{24}$$

By noticing that

$$\frac{\partial S^0}{\partial g_{ij}}\bigg|_{\mathfrak{g}(\mathcal{F})} = \frac{\delta \boldsymbol{S}}{\delta g_{ij}}\,, \tag{25}$$

we finally conclude that, under the assumption that the surface terms do not contribute, the right-hand side of (22) coincides with the Hamiltonian constraint in the canonical HJ form (19)

$$\partial_t \boldsymbol{S} = \int \mathrm{d}\boldsymbol{x}\left(\sqrt{-g}\;{}^3R + \frac{1}{\sqrt{-g}}\left(\frac{1}{2}g_{ij}g_{kl} - g_{ik}g_{il}\right)\frac{\delta \boldsymbol{S}}{\delta g_{ij}}\frac{\delta \boldsymbol{S}}{\delta g_{kl}}\right)\,. \tag{26}$$

Since the DWHJ theory reproduces solutions of the Einstein equations, the embedding conditions (21) imply that the Hamiltonian constraint is vanishing on the solutions, *i.e.* $\partial_t \boldsymbol{S} = 0$, and hence the timelessness of the canonical formalism of general relativity emerges from the DWHJ formulation as a consequence of the space-time splitting.

4. Conclusions

We derived the canonical functional derivative HJ equation from the partial derivative DWHJ equation for the scalar field theory in curved space-time and general relativity in metric variables. In both cases, the Ansatz proposed in [34], which relates the canonical HJ functional with the DW eikonal functions/densities, holds true. In general relativity, where we confined ourselves to the case of the Gaussian coordinates, we derived the standard Hamiltonian constraint in the HJ form from the DWHJ equation. We expect that a consideration in general coordinates will also reproduce the momentum constraint. The obtained results should be helpful for the comparison of canonical quantum gravity [26, 27] and precanonical quantization of general relativity [19–25], and for the study of the latter in the semiclassical approximation, where it should reproduce the DWHJ equation. They may also be helpful for understanding the origin of the problem of time in quantum gravity. We also expect that the DW and the DWHJ formulation of Einstein's equations can be useful for their numerical integration using the polysymplectic integrator which preserves the fundamental structure of the DW Hamiltonian form of field equations (*cf.* [37]). Our result can be viewed as a classical counterpart of the study of the relations between the functional Schrödinger representation in quantum field theory (see *e.g.* [36]) and the precanonical quantization based on the DW Hamiltonian theory, which has been undertaken in [12–18, 34, 38, 39] and whose extension to quantum gravity is so far unknown.

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