COSMOGRAPHIC RECONSTRUCTION TO DISCRIMINATE BETWEEN MODIFIED GRAVITY AND DARK ENERGY*

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Cosmography represents a model-independent approach potentially useful to discriminate among concurring cosmological scenarios. After reviewing the main features and shortcomings of standard cosmography, we highlight how to overcome the convergence issue jeopardizing current low redshift-cosmographic distances. To do so, we give particular attention to the use of cosmographic rational approximations, among them the Padé and Chebyshev polynomials. We thus focus on dark energy models and concurring extended and modified gravity models, in view of present cosmographic findings. We stress that current (and above all) future cosmographic constraints will be able to disentangle dark energy from alternative gravity, showing which model can be effectively reliable to describe the today observed accelerating universe.

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1. Introduction

Ever since the discovery of accelerated expansion [1], the description of the universe has been fundamentally modified from its original version. Observations immediately indicated that the universe is driven by a cosmological constant contribution that provides a negative pressure which acts to

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speed up the universe today. The most accepted paradigm which takes the cosmological constant into account is named ΛCDM model and consists in a matter term, composed by cold dark matter and baryons, with a cosmological constant contribution and a spatially flat geometry. Although appealing, the model suffers from several shortcomings, essentially related to the cosmological constant problem that can be split into a classical and quantum caveats, intimately related to the difficulties of matching the Standard Model of particle physics with cosmological observations. To overcome any issues, a simple possibility is to extend the cosmological constant hypothesis in favor of an evolving dark energy term [2]. Several other possibilities can be considered, *e.g.* the extended and/or modified theories of gravity which are capable of encompassing the ΛCDM issues, by means of first principles. In fact, the role played by vacuum energy is revised by means of corrections to the Einstein–Hilbert action [3]. This approach has reached great interest since it is possible to describe even the dark matter contribution within a single scheme based on the concept of geometrical fluids. Recently, the extended theories of gravity have taken novel consensus since Planck observations seem to indicate that inflationary phases can be driven by a simple $R+R^2$ potential, the Starobinsky model [4], in agreement with the simplest extension of Einstein's gravity known as $f(R)$ models [5]. Null diagnostics and model-independent techniques become essential to discriminate among models. Unfortunately, the lack of cosmic data and the difficulties in fitting high-redshift data plague severely the use of model-independent treatments. In other words, it is not yet possible to disentangle extended theories of gravity from dark energy models. However, the need of further investigations on model-independent strategies is essential to refine the current knowledge of universe dynamics. In this respect, cosmography represents a simple approach to model-independently handle the cosmological observables and to match them with data [6]. The idea is taking into account the cosmological principle and expanding the scale factor in Taylor series around our time t_0. Afterwards, all the other quantities of interest, above all cosmic distances and ladders, can be re-expressed in terms of such an expansion. The corresponding new approximated versions of these observables are functions of the derivatives of $a(t)$ and can be directly fitted with cosmic data.

In this work, we critically revise the state-of-the-art of cosmography, giving particular emphasis to the convergence problem and to the need of extending the standard kinematic approach through rational approximations such as Padé and Chebyshev polynomials. The paper is structured as follows. In Sec. 2, we analyze the role of cosmography and rational approximations. Afterwards, in Sec. 3, we compare our findings with extended and modified theories of gravity. Finally in Sec. 4, we discuss our conclusions and perspectives.

2. Cosmokinematics

Among all possible model-independent approaches, *cosmography* is likely the simplest one. It takes the observational assumption of the cosmological principle and it is based on the Taylor expansions of observables which can be directly compared with data. Thus, cosmography is, in principle, a powerful tool to break the degeneracy among cosmological models. The strategy is to expand $a(t)$ in Taylor series around the present time. The approach describes the universe kinematics considering only $a(t)$ derivatives. The $a(t)$ expansion is known as the *cosmographic series*, which provides, together with the *Hubble* definition $H(t) \equiv \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}t}$, the following quantities:

$$q(t) \equiv -\frac{1}{aH^2}\frac{\mathrm{d}^2 a}{\mathrm{d}t^2}\,, \qquad j(t) \equiv \frac{1}{aH^3}\frac{\mathrm{d}^3 a}{\mathrm{d}t^3}\,, \qquad s(t) \equiv \frac{1}{aH^4}\frac{\mathrm{d}^4 a}{\mathrm{d}t^4} \tag{1}$$

that are the *deceleration*, *jerk* and *snap* parameters. To discriminate among models, one needs that, at least up to s_0, the terms are univocally bounded at the 3σ confidence level. Unfortunately, we are still far from this due to the lack of cosmic data. This weakens dramatically the power of cosmography in disentangling effective dark energy models from any other extensions of Einstein's gravity.

The most important quantity one can expand is the luminosity distance

$$\begin{aligned} d_{\mathrm{L}}(z) \;=\; & \frac{1}{H_0}\left[z + \frac{1}{2}(1-q_0)z^2 - \frac{1}{6}\left(1 - q_0 - 3q_0^2 + j_0\right)z^3 \right. \\ & \left. + \frac{1}{24}\left(2 - 2q_0 - 15q_0^2 - 15q_0^3 + 5j_0 + 10q_0 j_0 s_0\right)z^4 + \mathcal{O}\left(z^5\right)\right], \end{aligned} \tag{2}$$

from which it is possible to obtain the Hubble expansion rate as $H(z) = \left[\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{d_{\mathrm{L}}(z)}{1+z}\right)\right]^{-1}$

$$H(z) \simeq H_0\left[1 + z(1+q_0) + \frac{z^2}{2}\left(j_0 - q_0^2\right) - \frac{z^3}{6}\left(-3q_0^2 - 3q_0^3 + j_0(3+4q_0) + s_0\right)\right]. \tag{3}$$

The main limitation of Eq. (2) is the inability of the currently available cosmological data to put tight constraints on the cosmographic parameters. Moreover, the kinematic expansion of the universe at early stages is not accounted due to the lack of a high-redshift formulation of cosmography. Every expansion is, in fact, plagued by two main issues: (1) arbitrary order of truncation which produces systematics within the numerical outcomes produced by experimental analyses; (2) limited predictability due to the fact that data exceed the limit of $z = 0$, *i.e.* the value around which one expands to obtain the cosmographic series.

The convergence of cosmographic series is therefore jeopardized by construction issues that can lead to poorly bound the set of coefficients of interest. To overcome this issue, one can consider rational approximations, such as Padé and Chebyshev polynomials. The method of Padé approximations is built up from the standard Taylor series of a generic function $f(z)$, through an (n,m) polynomial of the form of [7]

$$P_{n,m}(z) = \frac{\sum_{i=0}^{n} a_i z^i}{1+\sum_{j=1}^{m} b_j z^j} \tag{4}$$

whose Taylor expansion agrees with $\sum_i c_i f(z)$ to the highest possible order, *i.e.* $P_{n,m}(0) = f(0)$, $P'_{n,m}(0) = f'(0)$, $P_{n,m}^{(n+m)}(0) = f^{(n+m)}(0)$.

The Padé method still leaves a degree of subjectivity in the choice of the highest orders of expansion. Moreover, it works much better if one might approximate non-smooth functions where other numerical methods fail, *i.e.* in the cases of flexes or discontinuities in domains. So, conceptually, using Padé series to approximate well-defined cosmic distances may be seen as a non-suitable treatment in many cases.

Chebyshev polynomials may represent, on the contrary, alternatives to reduce systematics on fitted coefficients. The Chebyshev polynomials $T_n(z)$ are defined as

$$T_n(z) = \cos(n\theta)\,, \tag{5}$$

where $\theta = \arccos(z)$ and $n \in \mathcal{N}_0$. They are orthogonal polynomials with respect to the function $w(z) = (1-z^2)^{-1/2}$ for $|z| \leq 1$ such that $\int_{-1}^{1} T_n(z) T_m(z) w(z) = \pi$ if $n = m = 0$ and $\frac{\pi}{2}\delta_{nm}$ otherwise. To generate them, one can consider $T_{n+1}(z) = 2zT_n(z) - T_{n-1}(z)$, having $T_0(z) = 1$, $T_1(z) = z, T_2(z) = 2z^2 - 1, T_3(z) = 4z^3 - 3z, \ldots$ To account for the convergence issue, we can consider even rational versions of them [8]

$$R_{n,m}(z) = \frac{\sum_{i=0}^{n} a_i T_i(z)}{1+\sum_{j=1}^{m} b_j T_j(z)}\,. \tag{6}$$

This leads to $d_{\mathrm{L}}(z) = \frac{1}{H_0}\sum_{n=0}^{4} c_n T_n(z)$, where the terms c_n are $c_0 = \frac{1}{64}[18 + 5j_0(1+2q_0) - 3q_0(6+5q_0(1+q_0)) + s_0]$, $c_1 = \frac{1}{8}(7 - j_0 + q_0 + 3q_0^2)$, $c_2 = \frac{1}{48}[14 + 5j_0(1+2q_0) - q_0(14+15q_0(1+q_0)) + s_0]$ and so on.

3. Model-independent reconstruction of extended and modified theories of gravity

The above cosmographic analysis can be adopted to reconstruct dark energy models deriving the functional forms of Lagrangians from observational data. This method can be considered as a sort of *back-scattering* approach to the cosmological problem. A standard procedure in the $f(R)$ studies consists of assuming the gravity action and then finding out the dynamics by solving the modified Friedmann equations. The standard approach relies on postulating the form of $f(R)$ *a priori*, which determines the cosmological model. In what follows, instead, we present a model-independent method to reconstruct the functional form of the action [9]. In particular, the method of Taylor-expanding $f(R)$ for R approaching its late-time values is limited by the short range of redshift characteristic of observational data. Besides, the truncation of the Taylor polynomial reproducing the $f(R)$ function unavoidably introduces errors in the analysis. In this respect, the Padé polynomials may offer a possible solution to the convergence problem.

To apply our strategy, we first convert the time derivatives and the derivatives with respect to R into derivatives with respect to z according to the prescription

$$\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}t} = -(1+z)H\mathcal{F}_z\,, \tag{7}$$

$$\frac{\partial\mathcal{F}}{\partial R} = \frac{1}{6}\left[(1+z)H_z^2+H\left(-3H_z+(1+z)H_{zz}\right)\right]^{-1}\mathcal{F}_z\,, \tag{8}$$

where $\mathcal{F}(z)$ is an arbitrary function and we denote derivatives with respect to the redshift by the subscripts z. Then, after determining the values of the cosmographic parameters, one can combine the modified Friedmann equations, which provides us with the following second-order differential equation for $f(z)$ (see [6] for the details):

$$\begin{aligned}H^2 f_z &= \left[-(1+z)H_z^2+H\left(3H_z-(1+z)H_{zz}\right)\right]\\ &\times\Bigg[-6H_0^2(1+z)^3\Omega_{m0}-f-\frac{Hf_z\left(2H-(1+z)H_z\right)}{(1+z)H_z^2+H\left(-3H_z+(1+z)H_{zz}^2\right)}\\ &-\frac{f_{zz}\left((1+z)H_z^2+H(-3H_z+(1+z)H_{zz})\right)}{\left[(1+z)H_z^2+H\left(-3H_z+(1+z)H_{zz}\right)\right]^2}(1+z)H^2\\ &-\frac{(1+z)H^2\left(f_z\left(2H_z^2-3(1+z)H_zH_{zz}+H(2H_{zz}-(1+z)H_{zzz})\right)\right)}{\left[(1+z)H_z^2+H\left(-3H_z+(1+z)H_{zz}\right)\right]^2}\Bigg]\,.\end{aligned} \tag{9}$$

The initial conditions needed to solve the above equation are

$$f_0 = R_0+6H_0^2(\Omega_{m0}-1)\,, \tag{10}$$

$$f_z|_{z=0} = R_z|_{z=0}\,. \tag{11}$$

The analytical match over $f(z)$ can be approximated through three-parameter test-functions. Examples are:

$$
\begin{aligned}
\text{Exponential:} \qquad f_1(z) &= \mathcal{A}z + \mathcal{B}z^3 e^{\mathcal{C}z}\,, &(12)\\
f_2(z) &= \mathcal{A} + \mathcal{B}z^2 \sinh(1+\mathcal{C}z)\,, &(13)\\
f_3(z) &= \mathcal{A}z + \mathcal{B}z^3 \cosh(\mathcal{C}z)\,, &(14)\\
f_4(z) &= \mathcal{A}z^2 + \mathcal{B}z^4 \tanh(\mathcal{C}z)\,, &(15)\\
\text{Trigonometric:} \qquad f_5(z) &= \mathcal{A}z^3 + \mathcal{B}z^5 \sin(1+\mathcal{C}z)\,, &(16)\\
f_6(z) &= \mathcal{A}z^3 + \mathcal{B}z^4 \cos(1+\mathcal{C}z)\,, &(17)\\
f_7(z) &= \mathcal{A}z + \mathcal{B}z^2 \tan(\mathcal{C}z)\,, &(18)\\
\text{Logarithmic:} \qquad f_8(z) &= \mathcal{A}z + \mathcal{B}z^3 \ln(1+\mathcal{C}z)\,, &(19)
\end{aligned}
$$

where the set of coefficients is $(\mathcal{A},\ \mathcal{B},\ \mathcal{C})$. Finally, to determine $f(R)$, we need to invert the function $R(z)$ through a procedure that can be only done numerically due to the difficulties in inverting $H(z)$. Often it is requested, during the inversion, to relax the assumption $f'(R_0) = 1$. This leads to a $G_{\rm eff}$ slightly different from the gravitational constant G, within experimental limits. In the light of this, one gets

$$
\begin{aligned}
f_0 &= f'(R_0)\left(6H_0^2 + R_0\right) - 6H_0^2\Omega_{m0}\,, &(20)\\
f_z|_{z=0} &= f'(R_0)\ R_z|_{z=0}\,. &(21)
\end{aligned}
$$

In both cases, *i.e.* when $G_{\rm eff}$ is exactly the Newtonian constant or not, it is important to stress that the asymptotic value of $f'(R)$ depends on the accuracy of the cosmographic series at $z \gg 1$.

The extended theories of gravity can be therefore put in relation with cosmography by means of the aforementioned formalism. The same can happen for modified gravity models. Among all, let us consider how to reconstruct $f(T)$ functions of teleparallel gravity in a model-independent way, through cosmography. Knowing the set (H_0, q_0, j_0, s_0), we can again consider $f(T(z)) = f(z)$ and we can solve this formal equation numerically by combining the modified Friedmann equations with cosmic data. In doing so, we convert the derivatives with respect to time and the derivatives with respect to the torsion scalar into derivatives with respect to the redshift, getting a differential equation for $f(z)$

$$
\left(\frac{{\rm d}f}{{\rm d}z}\right)^{-1}\left[H(1+z)\frac{{\rm d}^2 f}{{\rm d}z^2} + 3f\frac{{\rm d}H}{{\rm d}z}\right] = \frac{1}{H}\left(\frac{{\rm d}H}{{\rm d}z}\right)^{-1}\left[3\frac{{\rm d}H}{{\rm d}z} + (1+z)\frac{{\rm d}^2 H}{{\rm d}z^2}\right]. \qquad (22)
$$

The form of $H(z)$ is the one imposed by cosmography, Eq. (3), driving our solutions by imposing the equivalence between the effective gravitation constant and the Newton constant

$$
\left.\frac{{\rm d}f}{{\rm d}z}\right|_{z=0} = 1\,, \qquad (23)
$$

and, moreover, to have the constraint

$$f(T(z=0)) = f(z=0) = 6{H_0}^2(\Omega_{m0} - 2)\,. \tag{24}$$

To this end, contrary to the case of $f(R)$ gravity, one can recast the cosmographic parameters as [10]

$$q_0 = -1 + \frac{3\tilde{\Omega}_{m0}}{2\left(1+2\tilde{F}_2\right)}\,, \tag{25}$$

$$j_0 = 1 - \frac{9\tilde{\Omega}_{m0}^2\left(3\tilde{F}_2+2\tilde{F}_3\right)}{2\left(1+2\tilde{F}_2\right)^3}\,, \tag{26}$$

$$s_0 = 1 - \frac{9\tilde{\Omega}_{m0}}{2\left(1+2\tilde{F}_2\right)} + \frac{45\tilde{\Omega}_{m0}^2\left(3\tilde{F}_2+2\tilde{F}_3\right)}{2\left(1+2\tilde{F}_2\right)^3} + \frac{27\tilde{\Omega}_{m0}^3\left(3\tilde{F}_2+12\tilde{F}_3+4\tilde{F}_4\right)}{4\left(1+2\tilde{F}_2\right)^4} - \frac{81\tilde{\Omega}_{m0}^3\left(3\tilde{F}_2+2\tilde{F}_3\right)^2}{2\left(1+2\tilde{F}_2\right)^5}\,, \tag{27}$$

where the unknown coefficients are formally rewritten as $\tilde{\Omega}_{m0} = \frac{\Omega_{m0}}{F_1}$, $\tilde{F}_i = \frac{F_i}{F_1}$ and $F_i = T_0^{i-1} f^{(i)}(T_0)$ for $i = 1,2,3,4$. Their exact expression can be found in [6]. To match the numerics, we can consider the same auxiliary functions as above and then we can reconstruct the function $f(T)$, once $H(z)$ is taken to be the cosmographic version of Eq. (3). To take into account the error propagation due to the uncertainties in cosmographic parameters, it is enough to baptize with a rescaling factor α the new auxiliary function, namely $\alpha f(z) \longrightarrow f(z)$. The value of the constant α will be determined from cosmological constraints. One thus gets

$$\begin{aligned} f(T) = \alpha\mathcal{A} + \frac{\alpha\mathcal{B}}{4\mathcal{Q}^2}\Bigg[& 2\left(q_0^2 - j_0\right) + \left(\frac{4\mathcal{M}(T)}{H_0^3}\right)^{1/3} + \left(\frac{16H_0^3}{\mathcal{M}(T)}\right)^{1/3} \\ & \times \left(j_0^2 + q_0^2\left(6+12q_0+7q_0^2\right) - 2j_0\left(3+7q_0+5q_0^2\right) - 2s_0(1+q_0)\right)\Bigg]^2 \\ \times \exp\Bigg\{ \frac{\mathcal{C}}{2\mathcal{Q}}\Bigg[& 2\left(q_0^2 - j_0\right) + \left(\frac{4\mathcal{M}(T)}{H_0^3}\right)^{1/3} + \left(\frac{16H_0^3}{\mathcal{M}(T)}\right)^{1/3} \\ & \times \left(j_0^2 + q_0^2\left(6+12q_0+7q_0^2\right) - 2j_0\left(3+7q_0+5q_0^2\right) - 2s_0(1+q_0)\right)\Bigg]\Bigg\}\,, \end{aligned} \tag{28}$$

where $\mathcal{M}(T), \mathcal{P}(T), \mathcal{Q}$ and $\mathcal{N}$ are explicitly reported in [6].

4. Final outlooks and perspectives

In this short report, we have analyzed the role of cosmography in the era of precision cosmology, and its implications in discriminating among concurring cosmological models. In particular, we studied whether cosmography can discriminate between dark energy scenarios and extended/modified theories of gravity. To do so, we initially reviewed the basic demands of the cosmographic treatments and its limitations. Afterwards, we analyzed the role played by rational approximations and their use in healing the convergence issue, *i.e.* the problem related to the use of high-redshift data in Taylor series expanded around $z = 0$. The matching of the cosmographic recipe in view of $f(R)$ and $f(T)$ theories has been therefore summarized. We gave particular emphasis on how to reconstruct, through a sort of back-scattering procedure, the shapes of $f(R)$ and $f(T)$ through the use of auxiliary functions $f(z)$. We proposed test-functions to be used in this scheme and we showed the main consequences in cosmology. We conclude that these approaches will need refined improvements to be fully-predictive. Indeed, until now, the cosmographic approach is essentially capable of suggesting the models that better adapt to kinematics, without windowing any new landscapes in the dark energy evolution. Even though new insights have been proposed by current cosmography, future steps to enable cosmography to disclose the nature of dark energy are essentially based on reformulating it in terms of high-redshift data. This will permit to handle any data with improved experimental bounds at a very significant statistical level.

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