COVARIANT FORMULATION OF LIGHT PROPAGATION IN COSMOLOGICAL MODELS*

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We present a covariant approach to the problem of light beam propagation in cosmological models within the framework of classical geometric optics in general relativity. Using the concept of screen surface orthogonal to the observer's world-line and to the bundle of geodesics, we introduce covariant four-dimensional definitions and derive propagation equations for Sachs and Jacobi optical fields and for the area distance.

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1. Introduction

The basics of the theory of light propagation were developed in [1, 2] where the geometry of congruence of null geodesics was considered. In this approach, there is used a pseudo-orthonormal tetrad which is parallelly transported along the rays, and the central role in equations is played by the optical scalars which characterize the rate of change of the geometry of cross section of the bundle. Another approach presented in [3–7] is based on the behavior of the connecting vectors which relate neighboring rays in the bundle and which obey the geodesic deviation equation. Both approaches are theoretically equivalent but computationally they provide two distinct ways for obtaining the area distance.

Here, we provide an alternative description of the light propagation which fully utilizes the notion of the observer's screen surface. It enables space-time tensor fields to be covariantly split into parts which are parallel to the observer's 4-velocity or to the beam's spatial direction or otherwise orthogonal to both of these vectors. This method is similar to the temporal-spatial splitting known from cosmology [8, 9].

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2. Formulation

We consider some cosmological model given by a metric field of space-time g_{mn} and a 4-velocity of cosmic fluid u_n. The 4-velocity is normalized as $u^a u_a = -1$. We assume the observer which is co-moving with the matter.

In the geometric approximation to the optics, the received electromagnetic waves are nearly plane, monochromatic and short. The light propagates along rays whose wave vector k_n is null, irrotational and obeys the geodesic equation

$$k^a k_a = 0\,, \qquad \nabla_m k_n = \nabla_n k_m\,, \qquad \dot{k}_n = 0\,, \tag{1}$$

where the dot denotes $\dot{X} \equiv k^a \nabla_a X$. This means that the light rays are potential null geodesics (the potential is the phase of the wave).

The measured circular frequency of the wave ω is defined as $\omega = -u^a k_a$. The screen field S_{mn} is defined as a symmetric field projecting onto the surface simultaneously orthogonal to the observer's 4-velocity and to the wave vector. These conditions yield

$$S_{mn} = -\frac{1}{\omega^2} k_m k_n + \frac{1}{\omega}\big(k_m u_n + u_m k_n\big) + g_{mn}\,. \tag{2}$$

For a given wave vector, all relevant quantities measurable by the observer are contained in the screen surface.

We also define the area field A_{mn} as a totally antisymmetric field on the screen surface

$$A_{mn} = -\frac{1}{\omega} k^b u^a A_{mnba}\,, \tag{3}$$

where A_{klmn} is the alternating, totally antisymmetric field of space-time. It represents the effective area element on the screen surface.

Let us now consider a light beam consisting of close geodesics. If the beam is infinitesimally narrow, all rays comprising this beam have the same wave vector k_n. The change rate of geometry of the beam's screen-section is described by the optical deformation rate field D_{mn}

$$D_{mn} = S_m{}^b S_n{}^a \nabla_b k_a\,. \tag{4}$$

This field is symmetric since the wave vector is irrotational. It could be further decomposed into its trace-free and pure-trace parts as

$$D_{mn} = \Sigma_{mn} + \frac{1}{2} S_{mn}\Theta\,, \qquad \Sigma^a{}_a = 0\,. \tag{5}$$

The traceless field Σ_{mn} is the optical shear rate and it represents the change rate of shape of the beam's screen-section. The scalar Θ is the optical expansion rate and it represents the change rate of size of the beam's screen-section. These two fields are called the Sachs optical fields.

The transport equations for optical fields along the considered beam are obtained from the Ricci identity for the wave vector

$$\nabla_l \nabla_m k_n - \nabla_m \nabla_l k_n = R_{lmn}{}^a k_a \,, \tag{6}$$

where R_{klmn} is the Riemann tensor. After suitable projections, we get two coupled equations

$$S_m{}^b S_n{}^a \dot{\Sigma}_{ba} = -\left(\Sigma_m{}^a \Sigma_{na} - \tfrac{1}{2} S_{mn} \Sigma^{ba} \Sigma_{ba}\right) - \Sigma_{mn} \Theta - S_m{}^d k^c S_n{}^b k^a C_{dcba} \,, \tag{7}$$

$$\dot{\Theta} = -\Sigma^{ba} \Sigma_{ba} - \tfrac{1}{2} \Theta^2 - k^b k^a R_{ba} \,, \tag{8}$$

where C_{klmn} is the Weyl tensor and R_{mn} is the Ricci tensor. These equations are called the Sachs optical equations. Since the optical expansion rate is singular at the observation event, we cannot solve this system of equations immediately.

The actual geometry of the beam's screen-section is characterized by the on-screen Jacobi field J_{mn} which is defined by the equations

$$S_m{}^b S_n{}^a \dot{J}_{ba} = D_m{}^a J_{an} \,, \tag{9}$$

$$u^a J_{na} = u^a J_{an} = 0 \,, \qquad k^a J_{na} = k^a J_{an} = 0 \,. \tag{10}$$

The Jacobi field represents the Jacobi matrix of the map relating the physical separations of rays within the beam with the angular separations of these rays seen on the observer's screen. The determinant of the Jacobi field J is the Jacobian of this map which is the ratio of the physical area of the beam's screen-section to its observed solid angle. Thus, we define the area distance Δ from the observer to the source as the square root of the determinant of the Jacobi field

$$\Delta^2 = J \,. \tag{11}$$

The determinant of the Jacobi field can be calculated with the help of the area field from

$$A_{mn} J = A^{ba} J_{mb} J_{na} \,, \tag{12}$$

which gives

$$J = \tfrac{1}{2} \left(J^b{}_b J^a{}_a - J^{ab} J_{ba} \right) \,. \tag{13}$$

This result establishes the covariant formula for the area distance.

The propagation equation for the Jacobi field is obtained from its definition by differentiation

$$S_m{}^b S_n{}^a \ddot{J}_{ba} = -S_m{}^d k^c S^{eb} k^a R_{dcba} J_{en} \,. \tag{14}$$

Once we find the solution of this equation, we can calculate the area distance.

Returning to the Sachs optical equations, it can be shown that the optical expansion rate is expressed by the area distance as follows:

$$\Theta = 2\frac{\dot{\Delta}}{\Delta}\,. \tag{15}$$

Hence, we rewrite the Sachs optical equations into a more suitable form

$$S_m{}^b S_n{}^a \dot{\Xi}_{ba} = -\frac{1}{\Delta^2}\left(\Xi_m{}^a \Xi_{na} - \frac{1}{2} S_{mn} \Xi^{ba} \Xi_{ba}\right) - \Delta^2 S_m{}^d k^c S_n{}^b k^a C_{dcba}\,, \tag{16}$$

$$\ddot{\Delta} = -\frac{1}{2}\frac{1}{\Delta^3}\Xi^{ba}\Xi_{ba} - \frac{1}{2}\Delta k^b k^a R_{ba}\,, \tag{17}$$

where we have introduced the scaled optical shear rate $\Xi_{mn} = \Delta^2 \Sigma_{mn}$. This system of equations can be solved to obtain the area distance directly.

In order to impose the initial conditions for the considered equations, one needs the relation between the scaled optical shear rate and the Jacobi field

$$\Xi_{mn} = -\Delta S_m{}^b S_n{}^a \left(J_{ba} \dot{\left(\frac{J^c{}_c}{\Delta}\right)} - J_b{}^c \dot{\left(\frac{J_{ca}}{\Delta}\right)} \right)\,. \tag{18}$$

Since the observation event is a vertex point for the beam's rays, the Jacobi field vanishes there

$$J_{mn}\big|_0 = 0\,. \tag{19}$$

By the relationships between the respective fields, this implies that

$$\Delta\big|_0 \;=\; 0\,, \qquad \frac{J_{mn}}{\Delta}\bigg|_0 = \frac{\dot{J}_{mn}}{\dot{\Delta}}\bigg|_0\,, \tag{20}$$

$$\frac{\Xi_{mn}}{\Delta^2}\bigg|_0 \;=\; 0\,, \qquad \Xi_{mn}\big|_0 = 0\,, \qquad \dot{\left(\frac{J_{mn}}{\Delta}\right)}\bigg|_0 = 0\,, \tag{21}$$

and additionally there follows the identity between initial conditions for derivatives of the area distance and the Jacobi field

$$\dot{\Delta}^2\big|_0 = \frac{1}{2}\left(\dot{J}^b{}_b \dot{J}^a{}_a - \dot{J}^{ab}\dot{J}_{ba}\right)\Big|_0\,. \tag{22}$$

In practice, we shall impose the initial conditions for only two of the components of the scaled optical shear rate. Likewise, we give the initial conditions for four of the components of the Jacobi field.

The initial condition for the derivative of the area distance comes from the physical requirement that in the vicinity of the vertex, the distance should correspond to the path traveled by the photon with respect to the observer. If S is the affine parameter along the geodesic x^n crossing the vertex, then the infinitesimal distance dl from the observer in the direction of the source can be estimated as

$$dl = -d_a \mathrm{d}x^a = -d_a k^a \mathrm{d}S = -\omega \mathrm{d}S\,. \tag{23}$$

Hence, this gives

$$\dot{\Delta}\big|_0 = -\omega\big|_0\,. \tag{24}$$

The initial conditions for the components of the derivative of the Jacobi field are subjected only to the identity mentioned above and otherwise, they are unrestricted.

3. Summary

We have presented the problem of light propagation in a narrow beam in terms of the splitting of space-time. This enabled us to give covariant definitions for basic quantities characterizing properties of the propagating light beam, most notably for the area distance. The formulation presented here is complementary to the existing two-dimensional approaches. Since it is mainly oriented on the observer's measurements, it could be especially useful in applications to cosmology for studies of light beam propagation in various cosmological models. More details can be find in [10].

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