ON THE PRECANONICAL STRUCTURE OF THE SCHRÖDINGER WAVE FUNCTIONAL IN CURVED SPACE-TIME*

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The functional Schrödinger equation in curved space-time is derived from the precanonical Schrödinger equation. The Schrödinger wave functional is expressed as the trace of the multidimensional product integral of precanonical wave function restricted to a field configuration. The functional Schrödinger representation of QFT in curved space-time appears as a singular limiting case of a formulation based on precanonical quantization, which leads to a hypercomplex generalization of quantum formalism in field theory.

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1. Introduction

Precanonical quantization [1–5] is the approach to field quantization based on the De Donder–Weyl (DW) generalization of Hamiltonian formalism to field theory [6] which does not require the space+time decomposition and treats all space-time variables on equal footing. Despite the DW theory has been known since the 1930s and it was considered as a possible basis of field quantization by Hermann Weyl himself [7], its various mathematical structures have been studied starting from the late 1960s (with the relevant notion of the multisymplectic structure coined in Poland [8]), it is the structure of the Poisson–Gerstenhaber algebra of Poisson brackets defined on differential forms found within the DW Hamiltonian formulation in [5, 9, 10] which has proven to be suitable for a new approach to field quantization. Further discussion of this bracket or its different treatments and generalizations can be found *e.g.* in [11–16]. It also has been instrumental in recent discussions of various classical field theoretic models of gravity and gauge theories in [17–19].

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Applications of precanonical quantization so far include quantum gravity in metric [20–23] and vielbein variables [24–28], and quantum Yang–Mills theory [29–31]. However, the connection with the standard techniques and concepts of QFT still remains insufficiently explored.

Many aspects of the relations between the DW Hamiltonian theory and the canonical Hamiltonian formalism have been studied since the early 1970s including [8, 10, 13, 14, 32–35]. The nature of those relations is that, typically, a covariant geometrical object from the DW Hamiltonian theory leads to its canonical counterpart after the space+time decomposition, restriction to a subspace representing the Cauchy data and then integration over it. Hence the name "precanonical" for the DW Hamiltonian formalism and the related quantization.

On the quantum level, the connection between precanonical quantization and the functional Schrödinger representation of QFT [36] was found for scalar fields [34, 37, 38] and YM fields [29–31] in flat space-time. As a step towards understanding the connections between precanonical quantization of gravity [20–24, 26–28] and the existing approaches based on canonical quantization [39, 40], we have also explored a possible extension of those results to curved space-time [41–43]. The present discussion is a concise presentation of those papers.

As in the case of flat space-time, it will be shown that the functional Schrödinger representation in curved space-times [44, 45] emerges from the description derived from precanonical quantization as a singular limiting case.

2. Precanonical description of quantum scalar theory

Let us start from the Lagrangian density $\mathfrak{L} = \frac{1}{2}\sqrt{g}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \sqrt{g}V(\phi)$, where $g_{\mu\nu}(x)$ is the space-time metric, $g := |\det(g_{\mu\nu})|$. It defines the densities of polymomenta $\mathfrak{p}^\mu := \frac{\partial\mathfrak{L}}{\partial\partial_\mu\phi} = \sqrt{g}g^{\mu\nu}\partial_\mu\phi$ and the DW Hamiltonian density $\mathfrak{H} = \sqrt{g}H := \mathfrak{p}^\mu\partial_\mu\phi(\mathfrak{p}) - \mathfrak{L} = \frac{1}{2\sqrt{g}}g_{\mu\nu}\mathfrak{p}^\mu\mathfrak{p}^\nu + \sqrt{g}V(\phi)$. Then the field equations take the DW Hamiltonian form of

$$\mathrm{d}\mathfrak{p}^\mu(x)/\mathrm{d}x^\mu = -\partial\mathfrak{H}/\partial\phi\,, \qquad \mathrm{d}\phi(x)/\mathrm{d}x^\mu = \partial\mathfrak{H}/\partial\mathfrak{p}^\mu\,. \tag{1}$$

The Poisson bracket operation defined by the weight $+1$ density valued x-dependent polysymplectic structure $\mathit{\Omega} = \mathrm{d}\mathfrak{p}^\mu\wedge\mathrm{d}\phi\wedge\varpi_\mu$, where $\varpi_\mu := \partial_\mu \lrcorner\ (\mathrm{d}x^0\wedge\mathrm{d}x^1\wedge\ldots\wedge\mathrm{d}x^{n-1})$ yields [1, 9, 10]

$$\{\![\mathfrak{p}^\mu\varpi_\mu,\phi]\!\} = 1\,, \qquad \{\![\mathfrak{p}^\mu\varpi_\mu,\phi\varpi_\nu]\!\} = \varpi_\nu\,, \qquad \{\![\mathfrak{p}^\mu,\phi\varpi_\nu]\!\} = \delta^\mu_\nu\,. \tag{2}$$

These fundamental Poisson brackets are quantized according to the modified Dirac quantization rule

$$[\hat{A}\,,\hat{B}] = -\mathrm{i}\hbar\sqrt{g}\widehat{\{[A,B]\}} \tag{3}$$

leading to the following representations of precanonical quantum operators:

$$\hat{\mathfrak{p}}^\mu = -\mathrm{i}\hbar\varkappa\sqrt{g}\gamma^\mu\partial_\phi\,, \qquad \hat{\varpi}_\nu = \frac{1}{\varkappa}\gamma_\mu\,, \qquad \widehat{H} = -\frac{1}{2}\hbar^2\varkappa^2\partial_{\phi\phi} + V(\phi)\,, \tag{4}$$

where γ^μ are x-dependent Dirac matrices, $\gamma^\mu\gamma^\mu + \gamma^\mu\gamma^\mu = 2g^{\mu\nu}$, the composition of operators is the symmetrized Clifford (matrix) product and $\varkappa$ is a ultraviolet scale appearing on dimensional grounds.

The curved space-time version of the precanonical Schrödinger equation takes the form of

$$\mathrm{i}\hbar\varkappa\gamma^\mu\nabla_\mu\Psi = \widehat{H}\Psi\,, \tag{5}$$

where $\nabla_\mu := \partial_\mu + \omega_\mu(x)$ is a covariant derivative of Clifford algebra valued wave functions $\Psi(\phi, x^\mu)$ with the spin connection matrix $\omega_\mu = \frac{1}{4}\omega_\mu^{AB}\underline{\gamma}_{AB}$ acting on the Clifford-valued Ψ via the commutator product, where $\underline{\gamma}_A$ are the Minkowski space Dirac matrices.

3. Relating the functional Schrödinger picture and precanonical quantization

The issue we would like to address here is a relation between the description of quantum fields derived from precanonical quantization and the standard QFT based on canonical quantization. More specifically, we would like to generalize the relation found in flat space-time in the case of quantum scalar field theory [34, 37, 38] and quantum YM theory [29, 31] to curved space-time. Our presentation here follows [41–43].

In curved space-time, quantum scalar field can be described in terms of the wave functional $\boldsymbol{\Psi}([\phi(\boldsymbol{x})], t)$ of field configurations $\phi(\boldsymbol{x})$ at the time t which obeys the Schrödinger equation [44, 45]

$$\mathrm{i}\hbar\partial_t\boldsymbol{\Psi} = \int \mathrm{d}\boldsymbol{x}\,\sqrt{g}\left(\frac{\hbar^2}{2}\;\frac{g_{00}}{g}\frac{\delta^2}{\delta\phi(\boldsymbol{x})^2} - \frac{1}{2}g^{ij}\partial_i\phi(\boldsymbol{x})\partial_j\phi(\boldsymbol{x}) + V(\phi)\right)\boldsymbol{\Psi}, \tag{6}$$

where the space-time coordinates adapted to the codimension-one space-like foliation are used such as $g_{0i} = 0$, g_{ij} is the induced metric on the space-like leaves of the foliation, and $x^\mu = (t, \boldsymbol{x}) = (t, x^i)$. As in all our papers, bold capital Greek letters like $\boldsymbol{\Psi}, \boldsymbol{\Phi}$ denote functionals and plain capital Greek letters like Ψ, Φ denote (precanonical) wave functions.

The problem of relating the canonical Schrödinger equation in functional derivatives with the partial derivative precanonical Schödinger equation (5) manifests itself already on the classical level: in [34], we have shown how the partial derivative DW Hamilton–Jacobi equation is related to the canonical

Hamilton–Jacobi equation in variational derivatives. Our initial consideration in flat space-time recently was extended to curved space-time and general relativity in [35].

On the quantum level, the idea is that the wave functional is a functional of field configurations because it is a composed functional of precanonical wave function $\Psi(\phi^a, x)$ restricted to the section Σ in the total space of the bundle with the coordinates (ϕ^a, x) which represents a field configuration $\phi(\boldsymbol{x})$ at the moment of time t: $\Psi_\Sigma := \Psi(\phi^a = \phi^a(\boldsymbol{x}), \boldsymbol{x}, t)$, *i.e.*

$$\boldsymbol{\Psi}([\phi(\boldsymbol{x})], t) = \boldsymbol{\Psi}[\Psi_\Sigma(\boldsymbol{x}, t), \phi(\boldsymbol{x})] \,. \tag{7}$$

In this case, the time dependence of the wave functional $\boldsymbol{\Psi}$ originates from the time dependence of precanonical wave function restricted to Σ, and the variational derivatives of $\boldsymbol{\Psi}$ can be related to the partial derivatives of Ψ_Σ. By denoting $\boldsymbol{\Phi}(\boldsymbol{x}) := \frac{\delta \boldsymbol{\Psi}}{\delta \Psi^{\mathrm{T}}_\Sigma(\boldsymbol{x})}$ (Ψ^{T} is the transpose of Ψ), we obtain

$$\begin{aligned}
\mathrm{i}\partial_t \boldsymbol{\Psi} &= \mathrm{Tr} \int \mathrm{d}\boldsymbol{x} \, \{\boldsymbol{\Phi}(\boldsymbol{x})\, \mathrm{i}\partial_t \Psi_\Sigma(\boldsymbol{x}, t)\} \,, \\
\frac{\delta \boldsymbol{\Psi}}{\delta \phi(\boldsymbol{x})} &= \mathrm{Tr}\, \{\boldsymbol{\Phi}(\boldsymbol{x}) \partial_\phi \Psi_\Sigma(\boldsymbol{x})\} + \frac{\bar{\delta} \boldsymbol{\Psi}}{\bar{\delta} \phi(\boldsymbol{x})} \,, \\
\frac{\delta^2 \boldsymbol{\Psi}}{\delta \phi(\boldsymbol{x})^2} &= \mathrm{Tr} \left\{ \delta(\mathbf{0}) \boldsymbol{\Phi}(\boldsymbol{x}) \partial_{\phi\phi} \Psi_\Sigma(\boldsymbol{x}) + 2 \frac{\bar{\delta} \boldsymbol{\Phi}(\boldsymbol{x})}{\bar{\delta} \phi(\boldsymbol{x})} \, \partial_\phi \Psi_\Sigma(\boldsymbol{x}) \right\} \\
&+ \mathrm{Tr}\,\mathrm{Tr} \left\{ \frac{\delta^2 \boldsymbol{\Psi}}{\delta \Psi^{\mathrm{T}}_\Sigma(\boldsymbol{x}) \otimes \delta \Psi^{\mathrm{T}}_\Sigma(\boldsymbol{x})} \, \partial_\phi \Psi_\Sigma(\boldsymbol{x}) \otimes \partial_\phi \Psi_\Sigma(\boldsymbol{x}) \right\} + \frac{\bar{\delta}^2 \boldsymbol{\Psi}}{\bar{\delta} \phi(\boldsymbol{x})^2} \,,
\end{aligned} \tag{8}$$

where $\partial_\phi \Psi_\Sigma(\boldsymbol{x}) := (\partial \Psi / \partial \phi)|_\Sigma(\boldsymbol{x})$, $\partial_{\phi\phi} \Psi_\Sigma(\boldsymbol{x}) := (\partial^2 \Psi / \partial \phi^2)|_\Sigma(\boldsymbol{x})$, $\bar{\delta}$ is the partial functional derivative with respect to $\phi(\boldsymbol{x})$, as opposite to the total functional derivative δ, and $\delta(\mathbf{0})$ is a regularized value of $\delta \Psi_\Sigma(\boldsymbol{x}) / \delta \Psi^{\mathrm{T}}_\Sigma(\boldsymbol{x}')$ at $\boldsymbol{x} = \boldsymbol{x}'$, which can be defined using a point splitting or lattice regularization to make sense of $(n-1)$-dimensional delta-function $\delta(\boldsymbol{x} - \boldsymbol{x}')$ at equal points.

The time derivative of Ψ_Σ is determined by the restriction of precanonical Schrödinger equation (5) in space+time decomposed form to Σ

$$\begin{aligned}
\mathrm{i}\partial_t \Psi_\Sigma &= -\mathrm{i}\gamma_0 \gamma^i \left(\frac{\mathrm{d}}{\mathrm{d}x^i} - \partial_i \phi(\boldsymbol{x}) \frac{\partial}{\partial \phi} \right) \Psi_\Sigma - \mathrm{i}\gamma_0 \gamma^i [\omega_i, \Psi_\Sigma] - \mathrm{i}[\omega_0, \Psi_\Sigma] \\
&\quad + \frac{\gamma_0}{\varkappa} \widehat{H} \Psi_\Sigma \,,
\end{aligned} \tag{9}$$

where $\frac{\mathrm{d}}{\mathrm{d}x^i} := \partial_i + \partial_i \phi(\boldsymbol{x}) \frac{\partial}{\partial \phi} + \partial_i \phi_k(\boldsymbol{x}) \frac{\partial}{\partial \phi_{,k}} + \ldots$ is the total derivative along Σ with ϕ_k denoting the fiber coordinates of the first-jet bundle of the bundle of field variables ϕ over space-time such that, when restricted to Σ, $\phi_k = \partial_k \phi(\boldsymbol{x})$.

By substituting (9) to (8), we obtain

$$
\begin{aligned}
\mathrm{i}\partial_t \boldsymbol{\Psi} \;=\; \operatorname{Tr}\int \mathrm{d}\boldsymbol{x}\, \Big\{ \boldsymbol{\Phi}(\boldsymbol{x},t)\Big(& \underbrace{-\mathrm{i}\gamma_0\gamma^i \frac{\mathrm{d}}{\mathrm{d}x^i}\Psi_\Sigma(\boldsymbol{x})}_{\mathrm{I}} + \underbrace{\mathrm{i}\gamma_0\gamma^i\partial_i\phi(\boldsymbol{x})\partial_\phi\Psi_\Sigma(\boldsymbol{x})}_{\mathrm{II}} \\
& \underbrace{-\,\mathrm{i}\gamma_0\gamma^i[\omega_i,\Psi_\Sigma(\boldsymbol{x})]}_{\mathrm{III}}\; \underbrace{-\,\mathrm{i}[\omega_0,\Psi_\Sigma]}_{\mathrm{IV}} \underbrace{-\frac{\varkappa}{2}\gamma_0\partial_{\phi\phi}\Psi_\Sigma(\boldsymbol{x})}_{\mathrm{V}} \\
& + \underbrace{\frac{1}{\varkappa}\gamma_0 V(\phi(\boldsymbol{x}))\Psi_\Sigma(\boldsymbol{x})}_{\mathrm{VI}}\Big)\Big\}\,.
\end{aligned}
\tag{10}
$$

By comparing the term V with the first term in (8) and (6), we conclude that the relation between them is only possible under the limiting mapping $\mapsto$ such that

$$\gamma_0\varkappa \mapsto \frac{g_{00}}{\sqrt{g}}\delta(\mathbf{0})\,. \tag{11}$$

We will see in what follows that the same condition also appears in other places when we are trying to establish a correspondence between the terms in (10) and the ones in (6).

The potential term VI in (10) should reproduce the potential term in (6). It is easy to see that it is possible only if at any spatial point $\boldsymbol{x}$, there is a mapping $\mapsto$ such that

$$\operatorname{Tr}\left\{\boldsymbol{\Phi}(\boldsymbol{x})\;\frac{1}{\varkappa}\gamma_0\Psi_\Sigma(\boldsymbol{x})\right\} \mapsto \sqrt{g}\;\boldsymbol{\Psi}\,. \tag{12}$$

The study of this conditions shows [41] that it can be fulfilled if the third term in (8) vanishes and the limiting condition (11) is satisfied.

Using the fact that $\sqrt{g}=\sqrt{g_{00}h}$, where $h := |\det(g_{ij})|$, and $\sqrt{g_{00}}\gamma^0 = \underline{\gamma}_0$, the condition (11) takes the form of

$$\underline{\gamma}_0\varkappa \mapsto \delta(\mathbf{0})/\sqrt{h} = \delta^{\mathrm{inv}}(\mathbf{0})\,, \tag{13}$$

where $\delta^{\mathrm{inv}}(\boldsymbol{x})$ is the invariant $(n-1)$-dimensional delta-function such that $\int \mathrm{d}\boldsymbol{x}\sqrt{h}\delta^{\mathrm{inv}}(\boldsymbol{x}) = 1$. Besides, the above definition of $\delta^{\mathrm{inv}}(\boldsymbol{x})$ implies that $\delta^{\mathrm{inv}}(\mathbf{0})$ is the inverse of the invariant volume element $\sqrt{h}\mathrm{d}\boldsymbol{x}$. Then (13) is equivalent to

$$\frac{1}{\varkappa} \mapsto \underline{\gamma}_0\sqrt{h}\mathrm{d}\boldsymbol{x}\,. \tag{14}$$

This interpretation will be used below when writing the product integral expressions of the wave functional in terms of precanonical wave function.

Our next observation is that the second term in (8) is similar to the term II in (10) in that both contain $\partial_\phi \mathit{\Psi}_\Sigma$ and do not have a counterpart in (6). Hence, they have to cancel each other at least in limit (11), *i.e.*

$$\mathrm{i}\boldsymbol{\Phi}(\boldsymbol{x})\gamma_0\gamma^i\partial_i\phi(\boldsymbol{x}) + \frac{g_{00}}{\sqrt{g}}\frac{\bar{\delta}\boldsymbol{\Phi}(\boldsymbol{x})}{\bar{\delta}\phi(\boldsymbol{x})} \mapsto 0\,. \tag{15}$$

The solution of (15) can be written in the form of

$$\boldsymbol{\Phi}(\boldsymbol{x}) = \boldsymbol{\Xi}([\mathit{\Psi}_\Sigma];\check{\boldsymbol{x}})\mathrm{e}^{-\mathrm{i}\phi(\boldsymbol{x})\gamma^i\partial_i\phi(\boldsymbol{x})/\varkappa}\,, \tag{16}$$

where the "integration constant" $\boldsymbol{\Xi}([\mathit{\Psi}_\Sigma];\check{\boldsymbol{x}})$ obeys $\frac{\bar{\delta}\boldsymbol{\Xi}([\mathit{\Psi}_\Sigma];\check{\boldsymbol{x}})}{\bar{\delta}\phi(\boldsymbol{x})} \equiv 0$. Therefore, the required cancellation of the terms with $\partial_\phi\mathit{\Psi}_\Sigma(\boldsymbol{x})$ in limit (11) fixes the form of the functional $\boldsymbol{\Phi}(\boldsymbol{x})$. This allows us to express the wave functional $\boldsymbol{\Psi}$ in the form of

$$\boldsymbol{\Psi} \sim \mathrm{Tr}\left\{\boldsymbol{\Xi}([\mathit{\Psi}_\Sigma];\check{\boldsymbol{x}})\ \mathrm{e}^{-\mathrm{i}\phi(\boldsymbol{x})\gamma^i\partial_i\phi(\boldsymbol{x})/\varkappa}\ \frac{\gamma_0}{\sqrt{g}\varkappa}\mathit{\Psi}_\Sigma(\boldsymbol{x})\right\}\Big|_{\varkappa\longmapsto\gamma_0\delta(\mathbf{0})/\sqrt{g}}\,, \tag{17}$$

which is valid at any point $\boldsymbol{x}$; $\sim$ here and in what follows is the equality up to a normalization factor which may depend on $\varkappa$ and $\sqrt{g}$. The notation $\{\dots\}\big|_{\varkappa\longmapsto\gamma_0\delta(\mathbf{0})/\sqrt{g}}$ means that $\varkappa$ is replaced by $\gamma_0\delta(\mathbf{0})/\sqrt{g}$ according to the limiting map (11).

Now, using (17), for the last term in (8), we get

$$\frac{1}{2}\frac{g_{00}}{\sqrt{g}}\frac{\bar{\delta}^2\boldsymbol{\Psi}}{\bar{\delta}\phi(\boldsymbol{x})^2} \mapsto -\frac{1}{2}\sqrt{g}g^{ij}\partial_i\phi(\boldsymbol{x})\partial_j\phi(\boldsymbol{x})\boldsymbol{\Psi} \tag{18}$$

that reproduces the second term in the functional derivative Schrödinger equation (6).

Thus, in the limiting case (11), we have derived all terms in (6) from (10). However, the terms I + III + IV in (10) still have played no role. One can argue based on the specific form of $\boldsymbol{\Phi}(\boldsymbol{x})$ found in (16) and the covariant Stokes theorem that I+III have vanishing contribution to the time evolution of $\boldsymbol{\Psi}$ (see [41] for details). Then the effective equation for $\partial_t\mathit{\Psi}_\Sigma$ in (8) reads

$$\begin{aligned}\mathrm{i}\partial_t\mathit{\Psi}_\Sigma \;&=\; \gamma_0\left(-\frac{\varkappa}{2}\partial_{\phi\phi} + \mathrm{i}\gamma^i\partial_i\phi(\boldsymbol{x})\partial_\phi + \frac{1}{\varkappa}V(\phi)\right)\mathit{\Psi}_\Sigma - \mathrm{i}[\omega_0,\mathit{\Psi}_\Sigma]\\ &=:\hat{H}_0 - \mathrm{i}[\omega_0,\mathit{\Psi}_\Sigma]\,.\end{aligned} \tag{19}$$

By introducing $\mathit{\Psi}'_\Sigma := U^{-1}\mathit{\Psi}_\Sigma U$, $\ \hat{H}'_0 := U^{-1}\hat{H}_0 U$, where

$$U(\boldsymbol{x},t) = \mathcal{T}\mathrm{e}^{-\int_0^t \mathrm{d}s\,\omega_0(\boldsymbol{x},s)} \tag{20}$$

is the tranformation determined by the time-ordered exponential, we obtain

$$
\begin{aligned}
\mathrm{i}\partial_t \Psi'_\Sigma &= U^{-1}\hat{H}_0 \Psi_\Sigma U = \hat{H}'_0 \Psi' \\
&= \gamma'_0 \left(-\frac{\varkappa}{2}\partial_{\phi\phi} + \mathrm{i}\gamma'^i \partial_i \phi(\boldsymbol{x})\partial_\phi + \frac{1}{\varkappa} V(\phi) \right) \Psi' ,
\end{aligned} \tag{21}
$$

where the transformation $\gamma^\mu \to \gamma'^\mu$: $\gamma'^\mu(x) := U^{-1}(x)\gamma^\mu(x)U(x)$ is an automorphism of the Clifford algebra as $\gamma'^\mu\gamma'^\nu + \gamma'^\nu\gamma'^\mu = 2U^{-1}g^{\mu\nu}U = 2g^{\mu\nu}$. Using (21), we write

$$
\mathrm{i}\partial_t \boldsymbol{\Psi} = \mathrm{Tr} \int \mathrm{d}\boldsymbol{x} \, \left\{ \frac{\delta \boldsymbol{\Psi}}{\delta \Psi'^{\mathrm{T}}_\Sigma(\boldsymbol{x})} \mathrm{i}\partial_t \Psi'_\Sigma \right\} = \mathrm{Tr} \int \mathrm{d}\boldsymbol{x} \, \left\{ \frac{\delta \boldsymbol{\Psi}}{\delta \Psi'^{\mathrm{T}}_\Sigma(\boldsymbol{x})} \hat{H}'_0 \Psi'_\Sigma \right\} . \tag{22}
$$

By comparing it with (8) and (9), we conclude that the term IV is taken into account if the quantities in the expression of the wave functional in (17) are replaced by the primed (U-transformed) ones, *i.e.*,

$$
\boldsymbol{\Psi} \sim \mathrm{Tr} \left\{ \boldsymbol{\Xi}'([\Psi'_\Sigma]; \check{\boldsymbol{x}}) \; \mathrm{e}^{-\mathrm{i}\phi(\boldsymbol{x})\gamma'^i\partial_i\phi(\boldsymbol{x})/\varkappa} \; \frac{\gamma'_0}{\sqrt{g}\varkappa} \Psi'_\Sigma(\boldsymbol{x}) \right\} \Big|_{\varkappa \longmapsto \gamma'_0 \delta(\mathbf{0})/\sqrt{g}} . \tag{23}
$$

The fact that this expression is valid for any choice of the point $\boldsymbol{x}$ allows us to fix $\boldsymbol{\Xi}'([\Psi'_\Sigma]; \check{\boldsymbol{x}})$ and obtain the expression of $\boldsymbol{\Psi}$ in the form of a formal continual product

$$
\boldsymbol{\Psi} \sim \mathrm{Tr} \left\{ \prod_{\boldsymbol{x}} \mathrm{e}^{-\mathrm{i}\phi(\boldsymbol{x})\gamma'^i\partial_i\phi(\boldsymbol{x})/\varkappa} \underline{\gamma_0} \Psi_\Sigma(\phi(\boldsymbol{x}), \boldsymbol{x}, t) \right\} \Big|_{\varkappa \mapsto \gamma'_0 \delta(\mathbf{0})/\sqrt{g}} . \tag{24}
$$

This symbolic formula expresses the Schrödinger wave functional in terms of precanonical wave function restricted to a field configuration Σ in general space-time (in the adapted coordinates with $g_{0i} = 0$).

The formal continual product expression in (24) can be understood as the multidimensional product integral [46, 47] with the invariant measure $\sqrt{h}\mathrm{d}\boldsymbol{x}$

$$
\boldsymbol{\Psi} \sim \mathrm{Tr} \left\{ \prod_{\boldsymbol{x}} \mathrm{e}^{-\mathrm{i}\phi(\boldsymbol{x})\gamma'^i(\boldsymbol{x})\partial_i\phi(\boldsymbol{x})/\varkappa} \underline{\gamma'_0} \Psi'_\Sigma(\phi(\boldsymbol{x}), \boldsymbol{x}, t) \right\} \Big|_{\frac{1}{\varkappa} \mapsto \underline{\gamma'_0} \sqrt{h} \mathrm{d}\boldsymbol{x}} , \tag{25}
$$

where the notation of the product integral of matrix-valued functions $F(\boldsymbol{x})$ is used

$$
\prod_{\boldsymbol{x}} \mathrm{e}^{F(\boldsymbol{x})\mathrm{d}\boldsymbol{x}} = \prod_{\boldsymbol{x}} \left(1 + F(\boldsymbol{x})\mathrm{d}\boldsymbol{x}\right) . \tag{26}
$$

Expression (25) generalizes our previous result in flat space-time [37] to curved space-times. In curved space-times, the spatial integration measure $\mathrm{d}\boldsymbol{x}$ is replaced by the invariant one $\sqrt{h}\mathrm{d}\boldsymbol{x}$, and the Dirac matrices are $\boldsymbol{x}$-dependent and nonlocally t-dependent if $\omega_0 \neq 0$.

Note that one-dimensional product integral coincides with the well-known path- or time-ordered exponential. The multidimensional generalization is more problematic [46, 47]. However, the trace of the multidimensional product integral can be understood as the continuum limit of the averaging of the product of matrices in the infinitesimal cells of space over all possible permutations of them in the product over all the cells, if the corresponding limit exists

$$\mathrm{Tr} \prod_{\boldsymbol{x}\in V} \mathrm{e}^{F(\boldsymbol{x})\mathrm{d}\boldsymbol{x}} := \lim_{N\to\infty} \frac{1}{N!} \mathrm{Tr} \sum_{P(N)} \mathrm{e}^{F(\boldsymbol{x}_1)\Delta\boldsymbol{x}_1} \mathrm{e}^{F(\boldsymbol{x}_2)\Delta\boldsymbol{x}_2} \ldots \mathrm{e}^{F(\boldsymbol{x}_N)\Delta\boldsymbol{x}x_N}, \tag{27}$$

where $P(N)$ is the set of all permutations of $(1, 2, \ldots, N)$, the volume of integration $V \ni \boldsymbol{x}$ is partitioned into N small sub-volumes $\Delta\boldsymbol{x}_1, \ldots, \Delta\boldsymbol{x}_N$ whose volumes are vanishing in the $N \to \infty$ limit, and $F(\boldsymbol{x}_i)$ denotes the matrix F at a point $\boldsymbol{x}_i \in \Delta\boldsymbol{x}_i$.

In static space-times when $\omega_0 = 0$, there is no non-local time dependence of the quantities in the expression of $\boldsymbol{\Psi}$ in terms of precanonical wave function, and equation (19) can be solved by the Ansatz

$$\Psi_\Sigma = \mathrm{e}^{+\frac{\mathrm{i}}{\varkappa}\phi(\boldsymbol{x})\gamma^i\partial_i\phi(\boldsymbol{x})} \Phi_\Sigma\,, \tag{28}$$

where Φ_Σ obeys

$$\mathrm{i}\partial_t\Phi_\Sigma = \widetilde{\gamma}_0(\boldsymbol{x}) \left(-\frac{\varkappa}{2}\partial_{\phi\phi} - \frac{1}{2\varkappa} g^{ij}(\boldsymbol{x})\partial_i\phi(\boldsymbol{x})\partial_j\phi(\boldsymbol{x}) + \frac{1}{\varkappa}V(\phi) \right) \Phi_\Sigma\,, \tag{29}$$

with

$$\widetilde{\gamma}^\mu(\boldsymbol{x}) := \mathrm{e}^{-\frac{\mathrm{i}}{\varkappa}\phi(\boldsymbol{x})\gamma^i\partial_i\phi(\boldsymbol{x})} \gamma^\mu(\boldsymbol{x}) \mathrm{e}^{+\frac{\mathrm{i}}{\varkappa}\phi(\boldsymbol{x})\gamma^i\partial_i\phi(\boldsymbol{x})} \tag{30}$$

and $\gamma^\mu \to \widetilde{\gamma}^\mu$ being a local Clifford algebra automorphism. From (29), it follows that Φ_Σ can be taken in the form of $\Phi_\Sigma = (1 + \underline{\gamma}^0)\Phi_\Sigma^\times$, where $\Phi_\Sigma^\times$ is a scalar function such that

$$\mathrm{i}\partial_t\Phi_\Sigma^\times = \sqrt{g_{00}} \left(-\frac{\varkappa}{2}\partial_{\phi\phi} - \frac{1}{2\varkappa} g^{ij}(\boldsymbol{x})\partial_i\phi(\boldsymbol{x})\partial_j\phi(\boldsymbol{x}) + \frac{1}{\varkappa}V(\phi) \right) \Phi_\Sigma^\times\,. \tag{31}$$

In terms of the scalar function $\Phi_\Sigma^\times$, the wave functional (25) reduces to the curvilinear product integral of scalar functions

$$\boldsymbol{\Psi} \sim \prod_{\boldsymbol{x}} \Phi_\Sigma^\times(\phi(\boldsymbol{x}), \boldsymbol{x}, t)\Big|_{\frac{1}{\varkappa}\mapsto\sqrt{h}\mathrm{d}\boldsymbol{x}}\,, \tag{32}$$

which can be defined without any complications related to the definition of the multidimensional product integral of non-commutative matrix functions.

4. Conclusion

It is demonstrated that in the case of scalar field theory in curved space-time, the precanonical Schrödinger equation (5) leads to the canonical functional derivative Schrödinger equation (6) in the limiting case when $\underline{\gamma}_0\varkappa$ is replaced by $\delta^{\mathrm{inv}}(\mathbf{0})$ whose regularized value is the ultraviolet cutoff of the volume of momentum space, whose introduction is one of the ways of defining the second variational derivative at equal points in the canonical functional derivative Schrödinger equation (6). As a by-product, we also obtain the expression of the Schrödinger wave functional as the continual product or product integral of precanonical wave function restricted to a field configuration. In space-times with vanishing zero-th component of spin connection, this expression reduces to the product integral of a scalar function obtained from precanonical wave function. In non-static space-times with non-vanishing zero-th component of spin connection, the Schrödinger wave functional is expressed in terms of the product integral of a non-local transformation of precanonical wave function defined by a time-ordered exponential of the zero-th component of spin connection.

We hope that these results can stimulate new approaches to the treatment of effects of quantum fields in curved space-time [48], such as the Hawking radiation, and they also can lead to better understanding of the nature of states of quantum fields in arbitrary space-times when a separation to positive- and negative-frequency modes is not possible. They also may help to clarify the connections between the existing approaches to quantum gravity originating from the canonical quantization of general relativity [39, 40] and precanonical quantization of gravity (see [20–28]).

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