NUCLEAR HIGH-RANK SYMMETRIES: FROM THE EARLY THEORY PREDICTIONS VIA TetraNuc COLLABORATION TO THE FINAL EXPERIMENTAL DISCOVERY*

J. Dudek[a,b], I. Dedes[b], J. Yang[b], A. Baran[b], D. Curien[a]
A. Gaamouci[c], A. Góźdź[b], A. Pędrak[d], H.L. Wang[e,a]

[a]Université de Strasbourg, IPHC, 23, rue du Loess, 67-037 Strasbourg, France
and CNRS, UMR7178, 67-037 Strasbourg, France
[b]Institute of Physics, Maria Curie-Skłodowska Univeristy
pl. Marii Curie-Skłodowskiej 1, 20-031 Lublin, Poland
[c]Laboratoire de Physique Théorique, Faculté de Physique
USTHB, BP 32, El Alia, 16-111 Bab Ezzouar, Algiers, Algeria
[d]National Centre for Nuclear Research, Pasteura 7, 02-093 Warszawa, Poland
[e]School of Physics and Engineering, Zhengzhou University
Zhengzhou 450001, China

(Received January 28, 2020)

A recent publication announcing the first identification of the tetrahedral and octahedral symmetries in subatomic physics — the symmetries often referred to as "high-rank" — is taken as an opportunity for a presentation of the series of turning points, which have lead to this discovery. It is known that the nuclear collective E2 (and E1) transitions vanish at the exact high-rank symmetry limit. Consequently, the first experimental tests aimed at studying the collective tetrahedral rotational bands with the deexciting transitions assumed very weak. At the same time, it has been assumed that the two symmetries will be broken, at least to an extent, and at least via the Coriolis angular momentum alignment and via the zero-point quadrupole motion around high-rank symmetric minima. Accordingly, the spin-parity sequences of the tetrahedral rotational bands were sought under the supposition that they resemble well-known octupole band properties. This strategy led to a few encouraging results but turned out to be inexact; the new strategy, based on the group and group-representation theories led finally to the evidence of signals from both tetrahedral and octahedral symmetries in one single nucleus: ^{152}Sm. Evolution covering nearly 25 years of this research is presented and the perspectives are discussed.

DOI:10.5506/APhysPolBSupp.13.419

* Presented at the XXVI Nuclear Physics Workshop *Key problems of nuclear physics*, Kazimierz Dolny, Poland, September 24–29, 2019.

1. Introduction

The recent discovery of the high-rank symmetries in subatomic physics, Ref. [1], provides an inspiration for advancing the studies of these symmetries to even broader extent. At the same time, it encourages a certain retrospective analysis of the evolution of the past research, which, we do hope, will become instructive when looking for the more and more efficient methods of the experimental identification of the two mentioned symmetries in many nuclei — but also possibly other point-group symmetries — throughout the Periodic Table.

The theoretical approach which has led to the discovery of the high-rank symmetries in subatomic physics turns out to be rather powerful more generally in studying nuclear geometrical properties. It employs combined techniques of the nuclear mean-field theory, group and group-representation theories. These methods belong to the most powerful tools used in quantum-mechanical description of the nuclear structure. The whole approach owes its successes mainly to its generality and the fact that the combination of the three techniques allows for constructing the experiment-comparable — and often unique — predictions and identification criteria.

The nuclear mean-field theory allows introducing a number of intuitive notions related to the nuclear 3D-geometry and, in particular, the geometry considerations in the context of the nuclear densities. Due to the short-range of the nucleon–nucleon interactions, these densities decrease fast to zero at the limits of their spatial distributions what allows introducing the fruitful concept of the nuclear surfaces. The mean-field potentials inherent to this theory have the spatial structure which follows the spatial structure of the densities — again due to the short range of nuclear interactions. Thus, studying the geometrical symmetries of the nuclear density distributions becomes strongly related to studying geometrical symmetries of the nuclear mean-field Hamiltonians. This latter step in turn allows for introducing the group theory as the most powerful tool in studying the mathematical consequences of groups of symmetry under which such Hamiltonians become invariant. As one of the consequences, we obtain access to exploiting the symmetry induced transition hindrance properties, the selection rules and the spectral rules imposed by the theory of irreducible representations of point-groups.

Among the most powerful techniques of the mean-field approaches, here within the self-consistent Hartree–Fock (Bogolyubov) category, one may cite the spin-parity, and the particle-number projection algorithms which allow for reproducing, among others, the unique rotational features of the nuclei with the densities (and shapes) invariant under any given symmetry point-group. For the discussion of the recent results obtained using these methods in the present context, the reader can be referred *e.g.* to Refs. [2–5]. An elementary description of the nuclear shapes and surfaces was in the literature conveniently formulated so far by employing the basis of the spherical harmonics, $\{Y_{\lambda\mu}\}$, and the corresponding expansion coefficients $\{\alpha_{\lambda,\mu}\}$ with the help of which an arbitrary nuclear surface, say $R(\vartheta, \varphi)$, can be practically expressed. Alternatively, the multipole-moment series, $\{Q_{\lambda\mu}\}$, provided by the self-consistent approaches, can be used as the source of an equivalent information.

We believe that a more efficient strategy in tracing back the variety of shapes and symmetries throughout the nuclear Periodic Table consists today in introducing several sequences of various point-group symmetries. Corresponding point-groups, sufficiently rich in symmetry elements (an example is provided by the octahedral group, O_h, and its tetrahedral sub-group T_d) can then be used to optimise the search for the possibly strongest deformed shell closures. Expressing equivalently, they allow to predict and study the largest gaps in the single-nucleon spectra and consequently — on average — the deepest local minima on the total potential-energy surfaces. This way of analysis is being followed up and the results will be published elsewhere.

In the following section, we present an overview of historical evolution of ideas and hypotheses which led to the discovery of high-rank symmetries. We focus on a selection of stepping stone experimental results which helped formulating identification techniques for the new symmetries in subatomic physics. In the rest of this article, we will discuss briefly the more recent suggestions based on the group representation theory which allow for obtaining the unique symmetry identification criteria.

2. Historical evolution of the high-rank symmetry search: TetraNuc Collaboration results and the follow-up

One of the first discussions addressing the hypothesis that the realistic nuclear mean-field may generate shape-isomeric states with the tetrahedral symmetry — drastically different from the prolate–spherical–oblate shape evolution and coexistence addressed most often in literature — can be found in Ref. [6]. To arrive at their hypothesis the authors performed calculations employing realistic deformed phenomenological Woods–Saxon Hamiltonian and its ‘universal’ parametrisation, *cf.* Ref. [7].

2.1. Early steps: Qualitatively new results and new language

The authors of Ref. [6] observe for the first time in the nuclear structure mean-field context the presence of the three families of nucleonic levels and, taking this into account, introduce the classification of the single-nucleon mean-field solutions in terms of the irreducible representations of the double tetrahedral group T_d^{D}. They employ as a guide-line an auxiliary spherical quantum-number labelling with the quantum numbers $\{n, \ell, j, m\}$ as follows:

$$E: \quad \left\{\left|n, \ell, j=\left(\ell+\tfrac{1}{2}\right), m=\tfrac{1}{2}, \tfrac{5}{2}, \tfrac{7}{2}, \tfrac{11}{2}, \tfrac{13}{2}, \ldots\right\rangle\right\}, \tag{1}$$

$$E^{*}: \quad \left\{\left|n, \ell, j=\left(\ell-\tfrac{1}{2}\right), m=\tfrac{1}{2}, \tfrac{5}{2}, \tfrac{7}{2}, \tfrac{11}{2}, \tfrac{13}{2}, \ldots\right\rangle\right\}, \tag{2}$$

which characterise the two 2-dimensional irreducible representations of the double group T_d^{D} and

$$G: \quad \left\{\left|n, \ell, j, m=\tfrac{3}{2}, \tfrac{9}{2}, \tfrac{15}{2}, \ldots\right\rangle\right\}, \tag{3}$$

the latter characterising the most original in the context 4-dimensional one. The corresponding labelling of the single-particle levels in terms of the irreducible representations of the tetrahedral double point group T_d^{D}, labelling which carries important nuclear structure symmetry messages has been introduced for the first time in place of the traditional Nilsson labelling.

In the same article, the single-particle spectral properties have been used to extract for the first time the *tetrahedral magic numbers* for the protons and similarly for the neutrons

$$Z_{T_d}^{\mathrm{magic}} = 56, 64, 70, 90 \quad \text{and} \quad N_{T_d}^{\mathrm{magic}} = 56, 64, 70, 90, 112, 136, 142\,. \tag{4}$$

The four-fold degeneracy of the nucleonic levels as well as the prediction of the new magic numbers together with that of the *new forms of the nuclear isomerism* attracted attention of many authors in the years to come. These were in particular the early Hartree–Fock studies of the tetrahedral symmetry effects in selected $Z = N$ nuclei in the vicinity of zirconium in Refs. [8] and [9] and next, by the ones in Ref. [10]. More systematic theoretical search of the tetrahedral symmetry in nuclei has been undertaken in Ref. [11] and followed by the one in Ref. [12], whereas a partial overview of the period of about 15 first years of the evolution can be found in Ref. [13].

Let us summarise this part of the discussion. The 'new language' involved at that time at least three concepts related to (a). Three families of nucleonic levels, (b). Tetrahedral magic numbers, and (c). A new class of isomers: Tetrahedral shape isomers.

2.2. Early period efforts: TetraNuc Collaboration projects

The first period of searching for experimental signs of the nuclear high-rank symmetries — inspired by the theory predictions — dates back to the beginning of this century. Theory and experimental efforts were coordinated within an informal international collaboration project *TetraNuc* (Tetrahedral Nuclei) supported by *Institut National de Physique Nucléaire et de Physique des Particules*, IN_2P_3, France, with one experimentalist and one theorist serving as spokespersons. Examples of the issues addressed by the collaboration can be found in Ref. [14].

The Collaboration proposed over a dozen of various experiments, some serving as the basis for the others. These efforts were contributed by over 120 scientists, among them over a dozen of theorists, from nearly 60 institutions all over the world. Members of the Collaboration performed experiments at large scale facilities such as: LNL, Legnaro, Italy; JYFL-JUROGAM, Jyväskylä, Finland; RIKEN, Japan; French–German Laue–Langevin installation at Grenoble, France; Gammasphere at Argonne NL, USA; INGA at TIFR, Mumbai, India — and, last but not least — important pilot projects run at the IPNO Tandem, Orsay, France. In total, members of the Collaboration published over 60 articles with theory and mixed experiment–theory profiles.

2.3. The first ideas in search for signs of tetrahedral symmetry

The physics ideas behind the mentioned projects were dominated by a few simplifying assumptions based on the knowledge of the "usual features" of nuclear collective rotational sequences, which can be found in data bases. Such sequences (bands) are characterised by the parabolic dependence of the energy *vs.* angular momentum, $E_I \propto I(I+1)$, with

$$I = I_{\text{b-h}}, (I_{\text{b-h}}+2), (I_{\text{b-h}}+4)\,, \ldots \tag{5}$$

where $I_{\text{b-h}}$ denotes the band-head spin which takes even or odd integer, alternatively half-integer values. The transitions connecting the consecutive levels are electric-quadrupole transitions with usually strong $B(\text{E2})$-values of a few dozens up to a couple of hundreds of Weisskopf units; of course, these transitions connect the levels of the same parity.

Vanishing collective transitions: High-rank symmetry bands of isomers

It was argued very early, on the basis of the symmetry-imposed hindrance properties, that the static electric quadrupole-, as well as dipole-moments vanish at the exact symmetry limit, Ref. [12]. At the same time, the spatial orientation of the tetrahedral, alternatively octahedral-symmetry nuclei — as non-spherical — can be defined, and one can expect that they produce rotational $E_I \propto I(I+1)$ energy *vs.* spin sequences. Therefore, in contrast to all the well-known electro-magnetic transition properties of the rotational quadrupole-deformed nuclei, the high-rank symmetry states should not be connected *via* any strong E2-transitions and, *e.g.* in the case of tetrahedral symmetry, are expected to produce isomer-bands with energies

$$T_d \text{ isomer-band energies}: \quad E_I^{\text{isomer}} = \frac{\hbar^2}{2\mathcal{J}_{\text{T}_\text{d}}}\, I(I+1)\,. \tag{6}$$

On the other hand, in realistic nuclear structure considerations, there exist well-established symmetry-breaking mechanisms. Let us limit ourselves to mentioning two of them: the *zero-point motion* and the *Coriolis alignment*. Since these symmetry breaking mechanisms are unavoidable in the nuclear structure context, the zero-point motion present always — whereas the Coriolis alignment — whenever the collective rotation is present, we refer to them as universal geometrical-symmetry breaking-mechanisms. They were expected to lead to some weak E2 and E1 collective (*i.e.* contributed by many nucleons) transitions and should have allowed using the powerful multi-detector γ-arrays for trying to trace back the corresponding signals of the presence of the high-rank symmetries.

Universal symmetry breaking mechanisms: Zero-point motion

Indeed, it is well-known from the collective model of Bohr that nuclear shapes are expected to oscillate around the shape at the equilibrium deformation. Thus, information about the static deformation at any tetrahedral symmetry minimum, *e.g.* the one characterised in the lowest-multipole order by α_{32}

$$T_d \text{ equilibrium shape}: \; \{\alpha_{\lambda\mu}\}_{\rm eq} \leftrightarrow \{\alpha_{20}=0, \alpha_{32}\neq 0\} \tag{7}$$

should be accompanied, among others, by some non-zero values of the quadrupole deformations, here referred to as “dynamical equilibrium” or most probable quadrupole deformations, $\{\alpha_{20}^{\rm dyn}\neq 0, \alpha_{22}^{\rm dyn}\neq 0\}$, consequently, generating some non-zero $B(\mathrm{E2})$-values, even if small or very small. This effect depends strongly on the flatness/steepness of the potential energy around the minimum and the behaviour of the deformation-dependent mass tensor, and is difficult to estimate reliably without any experimental countercheck information.

Universal symmetry breaking: Coriolis alignment

As the next universally present symmetry breaking element let us briefly discuss the Coriolis alignment of the individual-nucleonic angular momenta. It leads to distinguishing, in each rotating nucleus, a direction in space (the one of the total nuclear angular momentum) — thus breaking in particular the high-rank symmetries of interest here. At the early stage of searches for the experimental identification of the high-rank symmetries, it was, therefore, assumed that the sought tetrahedral bands[1] could be characterised by some weak E2-transitions at the moderately high-spins. The intensity of those transitions was *expected to decrease to zero* when the spins approached the band-head spin.

[1] These bands were assumed of negative parity due to their octupole (α_{32}) character and forming simplex-symmetry bands in analogy to the well-known pear-shape (α_{30}) octupole bands.

Focus on the band structures: Examining alternative hypotheses

It turned out that several low-lying negative-parity rotational bands with decreasing and vanishing E2-transitions exist in many nuclei with Z and N near the tetrahedral magic numbers. In fact, the low-spin E2-transitions in these negative-parity rotational-bands were sought in vain despite numerous attempts and using various and different ways of population of the bands in question. For instance in ${}^{156}_{64}\mathrm{Gd}_{92}$, a direct neighbour of the tetrahedral doubly-magic nucleus ${}^{154}_{64}\mathrm{Gd}_{90}$, such transitions were not seen below $I^\pi = 9^-$ state even though the decay scheme of that nucleus was studied using over 15 different reactions including β-, and *ec*-decay, (t,p), (p,t), (n,γ), (d,p), (p,d), (γ,γ'), (e,e'), (p,p'), (d,d') as well as a few neutron transfer reactions and the Coulomb excitation, *cf.* Ref. [15] for the early attempts. The lowest spin observed in these experiments in the band in question was $I^\pi = 3^-$.

Similar observations and difficulties with detection of the lowest E2-transitions in those bands apply to the neighbouring nuclei as for instance ${}^{152}_{64}\mathrm{Gd}_{88}$, ${}^{152}_{62}\mathrm{Sm}_{90}$ or ${}^{154}_{62}\mathrm{Sm}_{92}$, close neighbours of the just mentioned ${}^{156}_{64}\mathrm{Gd}_{92}$ and, at the same time, close neighbours of the tetrahedral doubly-magic configuration $(Z,N)_{T_d-\mathrm{magic}} = (64,90)$. Moreover, analogous results which demonstrate vanishing of the E2-transitions with the decreasing spin have been obtained in the actinide nuclei around the double-tetrahedral magic number combination of $(Z,N)_{T_d-\mathrm{magic}} = (90,142)$, namely, in ${}^{230}_{90}\mathrm{U}_{140}$ together with ${}^{232}_{90}\mathrm{U}_{142}$ and ${}^{234}_{90}\mathrm{U}_{144}$.

A possible alternative — with respect to the tetrahedral band interpretation attributed to the low-energy negative-parity bands discussed so far — was to return to the traditional way of thinking in terms of octupole-vibration rotational structures. In the present case, the vibrations would correspond to a non-zero quadrupole deformation minimum with the oscillations in the direction of either α_{30} (possible odd-spin $K^\pi = 0^-$ and/or 1^- vibrational bands) or in the direction of α_{32} (possible even-spin $K^\pi = 2^-$ and/or 3^- vibrational bands). The corresponding total energy surfaces are shown in Fig. 1.

The TetraNuc Collaboration performed a new test experiment using the JUROGAM γ-ray spectrometer at the University of Jyväskylä, employing the ${}^{154}(\alpha,2n){}^{156}$Gd reaction, Refs. [17] and [18]. The main aim was to collect the new information about the two negative parity bands, the odd-spin one for which the previously measured E2-transitions stop at $I^\pi = 9^-$ and the accompanying even-spin negative parity band interpreted so far as the simplex-partner of the previous one. The early studies of Refs. [19] and [20] aimed already at measuring the reduced transition probability ratios $B(\mathrm{E2})/B(\mathrm{E1})$ for the two bands. The results, even though limited to a few transitions only, indicated that the above ratios were on average about a factor of 50 higher for the even-spin band as compared to the odd-spin.

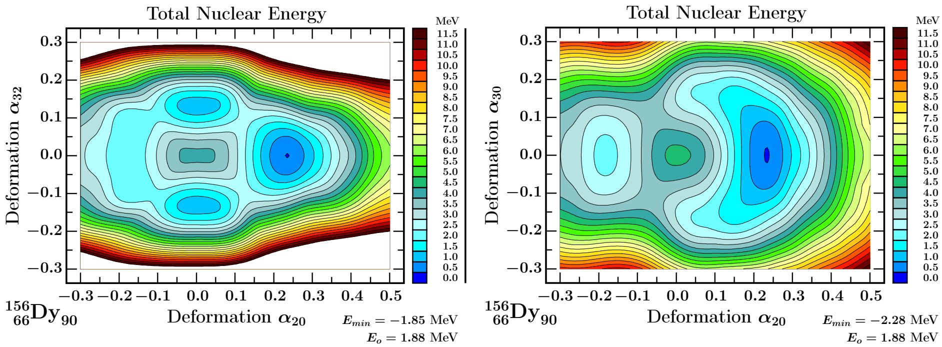

Fig. 1. Nuclear energies for ^{156}Dy projected on the α_{32} *vs.* α_{20} plane, after minimisation over octahedral deformation, left and on the α_{30} *vs.* α_{20} plane after minimisation over α_{40}, right.

Despite good overall instrumental performance achieved in this experiment, it was not possible to obtain any missing E2-transitions in the odd-spin negative-parity band in question, whereas $11^- \rightarrow 9^-$ transition cited previously in the literature was not reproduced. Moreover, the analysis of the transition intensities related to the even- and odd-spin bands established clearly that they significantly differ thus excluding the partnership hypothesis of the two bands and implying that if they could be attributed the octupole-vibration interpretation, the underlying octupole characters of the two are expected to be different. Such suggestions are supported by the comparison of the total energy surfaces in Fig. 1 showing that α_{30} pear-shape octupole vibrations, right-hand side, known to generate the induced dipole moments (thus lower $B(\mathrm{E2})/B(\mathrm{E1})$-ratios) should be accompanied by the α_{32}-type oscillations known not to induce dipole moments (thus generating possibly much higher $B(\mathrm{E2})/B(\mathrm{E1})$-ratios). The JUROGAM experiment confirmed the big disproportion of the two types of the $B(\mathrm{E2})/B(\mathrm{E1})$-ratios.

Chasing the possible false-positive high-rank symmetry signals

Let us return to the observations that in many nuclei in which vanishing E2-intensity mechanism expected on the basis of the tetrahedral-symmetry predictions in the vicinities of the tetrahedral doubly-magic configurations could be seen as overall strongly encouraging. However, let us emphasise that in each of the cited cases, the E2-decay within the band is accompanied by the E1-decay-out from the negative parity to the ground-state bands. Direct comparison of the corresponding results shows that whereas the quadrupole transition energies, $\Delta E_{\gamma-2}$, decrease linearly with spins [and the transition probabilities proportionally to $(\Delta E_{\gamma-2})^5$], E1-transition energies remain to a good approximation constant — and so do the E1-transition probabilities.

Therefore, the implied scenario may be that of the false-positive signal: The disappearance of the E2-transitions with decreasing spin at the bottom of the band may be a simple "kinematical" sign of the decreasing transition *energy* rather than the decrease in the *reduced transition probability*, B(E2), only the latter possibly due to the tetrahedral hindrance.

To verify which scenario is the appropriate one, we have employed the unprecedented instrumental possibilities at the Institute Laue–Langevin (ILL), Grenoble, France, Ref. [16]. Excited states in ^{156}Gd were populated in the ^{155}Gd$(n,\gamma)^{156}$Gd reaction and studied using the GAMS4/5 Bragg spectrometers. Gigantic cross section of $\sigma = 60\,900$ barn (compared to the five orders of magnitude smaller cross sections with heavy-ion fusion–evaporation processes) was one of the very unique features of this experiment. Another one: Thanks to the Bragg-diffusion mechanism, the γ-ray energies were measured with the unprecedented precision close to 1 eV.

Under these unusual, unique conditions the purpose was to measure the missing transition $5^- \to 3^-$ in the negative parity band. The second purpose was to measure life-time of the 5^- state using the line-shape Doppler analysis and deducing the corresponding B(E2) and the underlying Q_2-moment. This was achieved using the Gamma Ray Induced Recoil (GRID) method deducing the life-time from the Doppler broadening of the γ-line. The measured intensities and the life-time allowed deducing the quadrupole moment, $Q_2^{T_d=?}(5^-)$, of the "tetrahedral symmetry suspect" state 5^- and comparing it with the ground-state quadrupole moment

$$Q_2^{T_d=?}(5^-) = 7.1(2)\,b \quad \text{compared to} \quad Q_2^{\rm gs}\left(0^+\right) = 6.8(3)\,b\,, \tag{8}$$

suggesting that the two are, within experimental error bars, comparable.

Thus, the experimental results of Ref. [16] summarised here demonstrated that the vanishing of the E2-transitions at the bottoms of the tetrahedral-suspect bands is due to the competition between the E2 and E1 transitions and that in this sense, we were dealing with the false-positive signal. As a consequence, it became clear that other observables and techniques will be needed to establish the presence of high-rank symmetries in subatomic physics. This brings us to the most recent evolution, which has led recently to the experimental confirmation of both T_d and O_h symmetries. It will shortly be recalled next.

3. Changing strategy: Group-theory imposed structure of tetrahedral bands

Let us recall that the early attempts of identification of the high-rank symmetries were based on the strategical assumption that the symmetry breaking mechanisms, especially the one due to the zero-point motion, would impose band structures generally resembling the ones known from literature, the leading difference being the *relative weakness* of the collective E2 and E1 transitions. Since these originally proposed strategical assumptions underlying the principles of identification of the high-rank symmetries were invalidated by experimental results of the TetraNuc Collaboration — two leading examples of which were discussed briefly in the preceding section — it became clear that new ideas must be explored.

To follow this indication, a new strategy based on the adaptation of the point-group symmetry-based approaches similar to the ones employed in molecular physics has been followed within the Fukuoka–Strasbourg Collaboration, *cf.* Refs. [2–5] with the final discovery announcement in Ref. [1]. The reader is referred to the quoted references for details, whereas here, we wish to briefly present the main changes in the techniques advisable in the identification of the high-rank symmetries.

3.1. The case of even–even nuclei: T_d-symmetry rotational bands

The basic line in the new way of thinking, Ref. [1] and references therein, consists in exploring the symmetry properties of an abstract quantum rotor with the Hamiltonian invariant under — in the present case — T_d-symmetry. It is a textbook matter to demonstrate using point group representation theory that the corresponding rotor Hamiltonian can be analytically block-diagonalised, whereas each block corresponds to one among five, often denoted A_1, A_2, E, F_1 and F_2 (*cf. e.g.* Refs. [3, 4]) irreducible representations of the T_d-group. To illustrate the characteristic features of the new approach, it will be sufficient for the purposes of this article to limit the discussion to the tetrahedral symmetry ground-state band with the properties determined by irreducible representation A_1. The corresponding states are expected to form a common parabola composed of the following spin-parity sequence:

$$A_1: \quad 0^+,\, 3^-,\, 4^+,\, \underbrace{(6^+, 6^-)}_{\text{doublet}},\, 7^-,\, 8^+,\, \underbrace{(9^+, 9^-)}_{\text{doublet}},\, \underbrace{(10^+, 10^-)}_{\text{doublet}},\, 11^-,\, \underbrace{2\times 12^+,\, 12^-}_{\text{triplet}},\, \cdots \tag{9}$$

It needs to be emphasised that the corresponding sequence does not resemble at all the usual rotational band pattern with spins all either even or odd and of common parity. Just to the contrary, the high-rank symmetry rotational bands are characterised by a unique selection of spin-parity states several of them forming degenerate multiplets. Thanks to this "elaborate" structure, the identification of the experimental signals according to such selection criteria becomes unique.

3.2. The case of odd-A nuclei: $T_d^{\rm D}$-symmetry rotational bands

It follows from the group representation theory that the symmetries of the Hamiltonians describing spinors like nucleons or, more generally, half-integer spin objects like rotors of odd-A nuclei are described with the help of the spinor (so-called *double*) point groups, in the present case $T_d^{\rm D}$. The latter has three irreducible representations introduced already in Sect. 1, and denoted E, E^* and G. The first two of them can be recognised as parity-conjugate partners as it can be seen from the following two sequences:

$$E: \quad \tfrac{1}{2}^+, \tfrac{5}{2}^-, \underbrace{\left\{\tfrac{7}{2}^+, \tfrac{7}{2}^-\right\}}_{\text{doublet}}, \tfrac{9}{2}^+, \underbrace{\left\{\tfrac{11}{2}^+, \tfrac{11}{2}^-\right\}}_{\text{doublet}}, \underbrace{\left\{\tfrac{13}{2}^+, 2\times\tfrac{13}{2}^-\right\}}_{\text{triplet}}, \underbrace{\left\{\tfrac{15}{2}^+, \tfrac{15}{2}^-\right\}}_{\text{doublet}}, \cdots \tag{10}$$

$$E^*: \quad \tfrac{1}{2}^-, \tfrac{5}{2}^+, \underbrace{\left\{\tfrac{7}{2}^-, \tfrac{7}{2}^+\right\}}_{\text{doublet}}, \tfrac{9}{2}^-, \underbrace{\left\{\tfrac{11}{2}^-, \tfrac{11}{2}^+\right\}}_{\text{doublet}}, \underbrace{\left\{2\times\tfrac{13}{2}^+, \tfrac{13}{2}^-\right\}}_{\text{triplet}}, \underbrace{\left\{\tfrac{15}{2}^-, \tfrac{15}{2}^+\right\}}_{\text{doublet}}, \cdots \tag{11}$$

Whereas the detailed analyses of the properties of the ground-state tetrahedral-symmetry bands with odd-A will be a matter of case-by-case studies, an example of discussion of the involved properties can be found in Ref. [5]. It needs to be emphasised that, nevertheless, the band structures are unique and contain various combinations of opposite-parity states and multiplets.

4. Summary and conclusions

We took the recent experimental identification of the tetrahedral and octahedral (high-rank) symmetries in subatomic physics, Ref. [1], as an opportunity for briefly overviewing the 25 year history of the underlying research contributed by over 140 nuclear physicists from over 60 institutions all over the world forming an international collaboration project TetraNuc. We recall shortly the arrival starting point symmetry arguments and predictions, and remind the reader about the evolution of the basic ideas on how to identify these symmetries through the experimental data on real nuclei — down to the present status.

This work was partially supported by the National Science Centre, Poland (NCN) under contract No. 2016/21/B/ST2/01227 and the Polish–French COPIN-IN2P3 collaboration agreement under project numbers 04-113 and 05-119 and COPIGAL 2019.

REFERENCES

[1] J. Dudek *et al.*, *Phys. Rev. C* **97**, 021302(R) (2018).

[2] S. Tagami, Y.R. Shimizu, J. Dudek, *Prog. Theor. Phys. Suppl.* **196**, 334 (2012).

[3] S. Tagami, Y.R. Shimizu, J. Dudek, *Phys. Rev. C* **87**, 054306 (2013).

[4] S. Tagami, Y.R. Shimizu, J. Dudek, *J. Phys. G: Nucl. Part. Phys.* **42**, 015106 (2015).

[5] S. Tagami, Y.R. Shimizu, J. Dudek, *Phys. Rev. C* **98**, 024304 (2018).

[6] X. Li, J. Dudek, *Phys. Rev. C* **49**, R1246 (1994).

[7] The *Universal Woods–Saxon Hamiltonian* and associated, the so-called 'universal parametrisation' has been developed in a series of articles: J. Dudek, T. Werner, *J. Phys. G: Nucl. Phys.* **4**, 1543 (1978); J. Dudek *et al.*, *J. Phys. G: Nucl. Phys.* **5**, 1359 (1979); J. Dudek, W. Nazarewicz, T. Werner, *Nucl. Phys. A* **341**, 253 (1980); J. Dudek, Z. Szymański, T. Werner, *Phys. Rev. C* **23**, 920 (1981); and has been summarised in: S. Ćwiok *et al.*, *Comput. Phys. Commun.* **46**, 379 (1987). This approach is being used without modifications by many authors also today.

[8] S. Takami, K. Yabana, M. Matsuo, *Phys. Lett. B* **431**, 242 (1998).

[9] M. Yamagami, K. Matsuyanagi, M. Matsuo, *Nucl. Phys. A* **693**, 579 (2001).

[10] K. Zberecki, P.-H. Heenen, P. Magierski, *Phys. Rev. C* **79**, 014319 (2009).

[11] J. Dudek, A. Góźdź, N. Schunck, M. Miśkiewicz, *Phys. Rev. Lett.* **88**, 252502 (2002).

[12] J. Dudek *et al.*, *Phys. Rev. Lett.* **97**, 072501 (2006).

[13] J. Dudek *et al.*, *Acta Phys. Pol. B* **44**, 305 (2013).

[14] D. Curien, J. Dudek, K. Mazurek, *J. Phys.: Conf. Ser.* **205**, 012034 (2010).

[15] C.W. Reich, *Nucl. Data Sheets* **99**, 753 (2003).

[16] M. Jentschel *et al.*, *Phys. Rev. Lett.* **104**, 222502 (2010).

[17] Q.T. Doan *et al.*, *Acta Phys. Pol. B* **40**, 725 (2009).

[18] Q.T. Doan *et al.*, *Phys. Rev. C* **82**, 067306 (2010).

[19] J. Konijn *et al.*, *Nucl. Phys. A* **352**, 191 (1981).

[20] M. Sugawara *et al.*, *Nucl. Phys. A* **686**, 29 (2001).