Vol. 13 (2020) *Acta Physica Polonica B Proceedings Supplement* No 3

MODELS BASED ON TENSORS WITH RESPECT TO GROUPS*

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(Received January 29, 2020)

The paper contains the description of a new method for decomposition of arbitrary Hamiltonians and other operators into the tensor form. The method is based on a special kind of projection operators acting in the space of appropriate operators. The properties of these projection operators and the decomposition method are presented.

DOI:10.5506/APhysPolBSupp.13.465

1. Introduction

The idea of algebraic models which are presented in this paper has their origin in the generalized rotor model, where an expansion of a nuclear Hamiltonian in higher powers of intrinsic angular momenta is performed [1, 2]. However, the rotational degrees of freedom are usually insufficient to simulate spectra and transitions properly. The interesting and promising question at this point is an extension of the generalised rotor model onto groups other than the rotation group SO(3) [3].

* Presented at the XXVI Nuclear Physics Workshop *Key problems of nuclear physics*, Kazimierz Dolny, Poland, September 24–29, 2019.

The Hamiltonian in these models is expressed as a linear combination of tensors with respect to a given group G. Such a form of the Hamiltonian has a lot of benefits. It allows to introduce in an easy way required symmetries of the system under consideration (if the symmetry group is contained in G). It also allows for modeling of the external as well as the intrinsic symmetries [4, 5]. By using either microscopic methods or simply fitting procedure (due to free parameters), it gives a low-cost method for reproducing essential properties of nuclei.

Well-defined transformation properties of tensors with respect to a group is a very strong condition, which provides clear and straightforward physical interpretation. It also extends the range of problems which can be solved analytically. This opens the possibility to consider more complicated, microscopic models, on the other hand, it helps also in more effective designing and controlling numerical calculations. That is why better understanding of the mathematical structure of this operators is valuable from the physical point of view.

In the first part of the paper, we shortly introduce a construction of the Hamiltonians in algebraic models as a sum of tensors with respect to an arbitrary compact group G. At the next part, we introduce projection operators which act on the space of operators. The method of decomposition of an arbitrary operator into a sum of tensors with respect to this compact group is introduced. In the last part, the proof of existence of decomposition of an arbitrary operator into a sum of tensors with respect to a given compact group is shown.

2. Construction of models

The main idea of the models based on tensors with respect to groups is the existence of the description of Hamiltonians and observables as linear combinations (more generally functions) of tensors with respect to a given group G. The building blocks, in this case, are generators $\hat{X}_\zeta$ of the group G written in the tensor form

$$\hat{T}^{\mathit{\Gamma}}_{\zeta} = \left(\left(\left(\hat{X}\otimes\hat{X}\right)\otimes\hat{X}\right)\otimes\ldots\otimes\hat{X}\right)^{\mathit{\Gamma}}_{\zeta}\,, \quad \hat{g}\,\hat{T}^{\lambda}_{\zeta}\,\hat{g}^{-1} = \sum_{\eta}\mathit{\Delta}^{\lambda *}_{\eta\zeta}(g)\,\hat{T}^{\lambda}_{\eta}\,, \quad g\in\mathrm{G}\,, \tag{1}$$

where

$$\left(\hat{T}^{\mathit{\Gamma}_1}\otimes\hat{T}^{\mathit{\Gamma}_2}\right)^{\mathit{\Gamma}}_{\zeta} = \sum_{\zeta_1\zeta_2}\left(\mathit{\Gamma}_1\zeta_1\mathit{\Gamma}_2\zeta_2|\mathit{\Gamma}\zeta\right)\hat{T}^{\mathit{\Gamma}_1}_{\zeta_1}\,\hat{T}^{\mathit{\Gamma}_2}_{\zeta_2}\,. \tag{2}$$

The space of states is $\mathcal{K} = L^2(\mathrm{G}, \mathrm{d}\mu(\theta))$, where $\mathrm{d}\mu(\theta)$ is the Haar measure on the group G.

This structure allows for introducing the intrinsic degrees of freedom. In this case, the building blocks are generators $(\hat{X}_\zeta, \hat{\bar{X}}_\zeta)$ of the partner groups $\mathrm{G} \times \overline{\mathrm{G}}$.

This form of the building blocks offers more flexibility in the construction of the operators with given symmetry or transformation properties which are required for typical observables (electromagnetic transition operators, tensor interactions, *etc*). It does not apply only to transformation properties with respect to the group G but also to every subgroup $\tilde{\mathrm{G}} \subset \mathrm{G}$.

An integral part of this model are the bases of appropriate irreducible representations of the group G. Using such bases, one can extremely simplify the required numerical calculations. For example, if the Hamiltonian is invariant with respect to the group, the eigenproblem can be solved separately for every irreducible representation. It significantly reduces the time of calculations and dimension of the problem.

This form of the algebraic models gives also the possibility to build phenomenological Hamiltonians. If the Hamiltonian is expressed as a linear combination of group tensors,

$$\hat{H} = \sum_{\Gamma\mu} h^{\Gamma}_{\mu} \, \hat{T}^{\Gamma}_{\mu} \,,$$

the coefficients in the linear combination can be calculated or fitted to experimental data.

3. Projection operators defined on space of operators

In this part, we introduce decomposition of an arbitrary operator into a sum of tensor operators with respect to a given compact group. In this method, we use some special projection operators based on the irreducible representation of the group G. This operators were constructed in analogy to the generalized projection operators introduced for example in [6].

Let G be a compact group acting on a linear space, for $g \in G$, $\hat{g} : \mathcal{K} \to \mathcal{K}$, $\hat{H} : \mathcal{K} \to \mathcal{K}$. Let us define the following operator:

$$\hat{P}^{\Gamma}_{\eta\nu}\left(\hat{H}\right) = d_{\Gamma} \int_{G} \mathrm{d}\mu(g) \; \Delta^{\Gamma *}_{\eta\nu}(g) \; \hat{g} \, \hat{H} \, \hat{g}^{-1} \,, \tag{3}$$

where $\Delta^{\Gamma *}_{ab}(g)$ is an irreducible representation of the group G, d_{Γ} is the dimension of the representation Γ and $d\mu(g)$ is the Haar measure on G.

For any application, it is interesting to know the transformations properties of the operators $\hat{P}^{\Gamma}_{\eta\nu}(\hat{H})$ with respect to the action of the group elements

$$\hat{g} \, \hat{P}^{\Gamma}_{\eta\nu}\left(\hat{H}\right) \hat{g}^{-1} = \sum_{\mu} \Delta^{\Gamma}_{\mu\eta}(g) \; \hat{P}^{\Gamma}_{\mu\nu}\left(\hat{H}\right) \,. \tag{4}$$

Operators $\hat{P}^{\Gamma}_{\eta\nu}(\hat{H})$ form a tensor of the rank Γ with respect to the group independently of the transformation properties of the operator $\hat{H}$. Composition of two operators $\hat{P}^{\Gamma}_{\eta\nu}(\hat{H})$ gives

$$\hat{P}^{\Gamma_1}_{\eta_1\nu_1}\left(\hat{P}^{\Gamma_2}_{\eta_2\nu_2}\left(\hat{H}\right)\right)=\delta_{\Gamma_1\Gamma_2}\,\delta_{\nu_1\eta_2}\,\hat{P}^{\Gamma_1}_{\eta_1\nu_2}\left(\hat{H}\right)\,. \tag{5}$$

By using the above property, one can construct these generalized projection operators in the following way:

$$\hat{P}^{\Gamma}_{\eta}\left(\hat{H}\right)=\hat{P}^{\Gamma}_{\eta\eta}\left(\hat{H}\right)\,. \tag{6}$$

Operators (6) can form a specific resolution of unity

$$\hat{H}=\sum_{\Gamma\eta}\hat{P}^{\Gamma}_{\eta}\left(\hat{H}\right)\,. \tag{7}$$

One can prove the above equation by comparing matrix elements of operators at both sides in the basis of functions of irreducible representations of the group G

$$\begin{aligned}
&\left\langle\kappa'\Lambda'\mu'\right|\sum_{\Gamma\eta}d_{\Gamma}\int d\mu(g)\Delta^{\Gamma*}_{\eta\eta}(g)\hat{g}\hat{H}\hat{g}^{-1}\left|\kappa\Lambda\mu\right\rangle\\
&=\sum_{\Gamma\eta}d_{\Gamma}\int d\mu(g)\Delta^{\Gamma*}_{\eta\eta}(g)\sum_{\tilde{\mu}'}\Delta^{\Lambda'}_{\tilde{\mu}'\mu'}(g)\sum_{\tilde{\mu}}\Delta^{\Lambda}_{\mu\tilde{\mu}}(g)\left\langle\kappa'\Lambda'\tilde{\mu}'\left|\hat{H}\right|\kappa\Lambda\tilde{\mu}\right\rangle\\
&=\sum_{\Gamma\eta\tilde{\mu}'\tilde{\mu}}\left(\Lambda'\tilde{\mu}'\Lambda\mu|\Gamma\eta\right)\left(\Lambda'\mu'\Lambda\tilde{\mu}|\Gamma\eta\right)\left\langle\kappa'\Lambda'\tilde{\mu}'\left|\hat{H}\right|\kappa\Lambda\tilde{\mu}\right\rangle\\
&=\sum_{\tilde{\mu}'\tilde{\mu}}\delta_{\tilde{\mu}'\mu'}\delta_{\tilde{\mu}\mu}\left\langle\kappa'\Lambda'\tilde{\mu}'\left|\hat{H}\right|\kappa\Lambda\tilde{\mu}\right\rangle=\left\langle\kappa'\Lambda'\mu'\left|\hat{H}\right|\kappa\Lambda\mu\right\rangle\,.
\end{aligned} \tag{8}$$

This result shows that by using operators $\hat{P}^{\Gamma}_{\eta}$ every operator can be expressed as a sum of tensor operators with respect to the group G.

It is not an orthogonal decomposition in the sense of $\hat{P}^{\Gamma}_{\eta}(H)\hat{P}^{\Gamma}_{\mu}(H)\neq 0$ if $\mu\neq\eta$.

Such operators can be helpful in the analysis of symmetric substructures of a given Hamiltonian. For example, if one expects that the Hamiltonian possesses symmetry with respect to a given group $\tilde{\mathrm{G}}$ which is broken by a small unknown term, one can restore a symmetric part by using the projection operator $P^{\Gamma_0}_0$, where Γ_0 is a scalar representation of the group $\tilde{\mathrm{G}}$.

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